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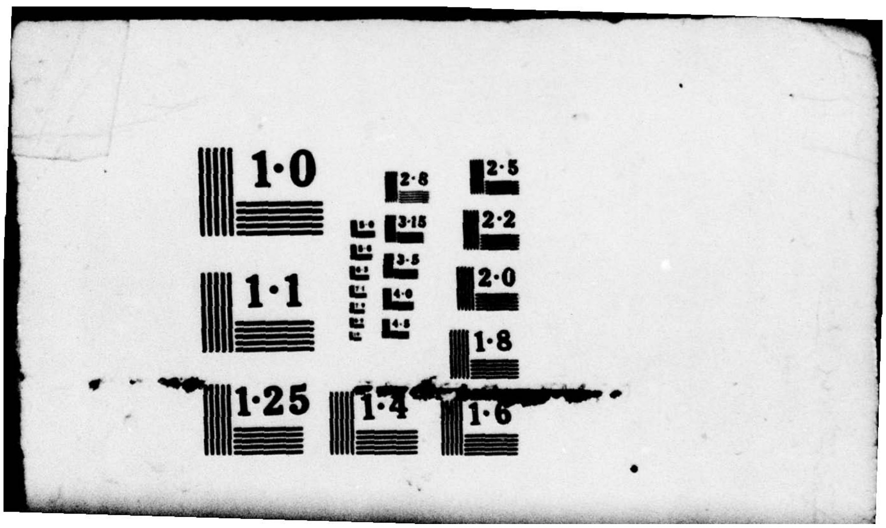
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FURTHER RATIONAL APPROXIMATIONS FOR THE INCOMPLETE GAMMA FUNCTION

LEVEL

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July 31, 1963

Contract No. Nonr-2638(00)(X)

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For

Applied Mathematics Laboratory
David W. Taylor Model Basin
Washington, D.C.

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FURTHER RATIONAL APPROXIMATIONS FOR THE INCOMPLETE GAMMA FUNCTION

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by

10 Yudell L. Luke
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PREFACE

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Y.L.L. and W.F.

Approved for:

MIDWEST RESEARCH INSTITUTE

Sheldon L. Levy, Director
Mathematics and Physics Division

August 5, 1963

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INTRODUCTION

In a previous report [1] we studied rational approximations to the incomplete gamma function. These were based on the asymptotic expansion of the latter function. In the present study, a similar analysis is done for the same function based on one of its ascending series representations. The work is related to other ~~the~~ developments ~~in [2] and [3]~~, but there are numerous improvements and some new features.

I. THE INCOMPLETE GAMMA FUNCTION

We consider

$$\gamma(z,\nu) = \int_0^z t^{\nu-1}e^{-t}dt \ , \ R(\nu) > 0 \ , \tag{1.1}$$

$$\Gamma(z,\nu) = \Gamma(\nu) - \gamma(z,\nu) \tag{1.2}$$

where

$$\Gamma(z,\nu) = \int_z^{\infty e^{i\omega}} t^{\nu-1}e^{-t}dt \ , \ z = x+iy \ , \ x \ , \ y \ \text{ and } \ \omega \ \text{ are real} \ ,$$

$$|\omega| < \pi/2 \ , \ R(\nu) > 0 \ ; \ |\omega| = \pi/2 \ , \ 0 < R(\nu) < 1 \ . \tag{1.3}$$

In (1.3) the path of integration lies in the z plane cut along the negative real axis and is the ray $\eta \exp(i\omega)$, $\eta \to \infty$ except for an initial finite path. If $z \neq 0$, $|\omega| < \pi/2$, the integral in (1.3) exists without the restriction on ν .

The connection of these functions to the confluent hypergeometric functions is given by

$$\gamma(z,\nu) = \nu^{-1}z^{\nu}e^{-z}\Phi(1,1+\nu;z) = \nu^{-1}z^{\nu}\Phi(\nu,1+\nu;-z) \ , \tag{1.4}$$

$$\Phi(a,c;z) = {}_1F_1(a;c;z) \quad , \tag{1.5}$$

$$\Gamma(z,\nu) = z^{\nu}e^{-z}\psi(1,1+\nu;z) = e^{-z}\psi(1-\nu,1-\nu;z) \quad , \tag{1.6}$$

$$\psi(a,c;z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\,\Phi(a,c;z) + \frac{\Gamma(c-1)}{\Gamma(a)}\,z^{1-c}\Phi(a-c+1,2-c;z) \quad . \tag{1.7}$$

The statement (1.4) is

$$\gamma(z,\nu) = z^{\nu}e^{-z}\sum_{k=0}^{\infty}\frac{z^k}{(\nu)_{k+1}} = z^{\nu}\sum_{k=0}^{\infty}\frac{(-)^k z^k}{k!(\nu+k)} \quad . \tag{1.8}$$

In the above formulas, standard generalized hypergeometric notation is used. Thus

$${}_pF_q\left(\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_q\end{matrix}\middle| z\right) = {}_pF_q(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;z)$$

$$= \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_k z^k}{\prod_{i=1}^{q}(b_i)_k k!}\ , \quad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \quad . \tag{1.9}$$

For further discussion of the hypergeometric functions, confluent hypergeometric functions and the incomplete gamma function, see [4, Chs. 2,4,6], [5, Ch. 9], [6], [7], and [8].

II. RATIONAL APPROXIMATIONS TO $\gamma(ze^{i\pi},\nu)$

In [2] we proved that

$$\nu z^{-\nu} e^{-z-i\pi\nu}\gamma(ze^{i\pi},\nu) = V_n(z,\nu) + R_n(z,\nu) \quad ,$$

$$V_n(z,\nu) = \frac{A_n(z,\nu)}{B_n(z,\nu)} \; , \; R_n(z,\nu) = \frac{P_n(z,\nu)}{B_n(z,\nu)} \quad , \tag{2.1}$$

where

$$A_n(z,\nu) = \sum_{k=0}^{n} \frac{(-n)_k(n+\nu+1)_k}{(\nu+1)_k k!} z^n \; {}_3F_1\left(\begin{matrix}-n+k, n+\nu+1+k, 1\\ 1+k\end{matrix}\middle| -1/z\right) \quad , \tag{2.2}$$

and $B_n(z,\nu)$ is the $k = 0$ term in (2.2). In this event, the ${}_3F_1$ becomes a ${}_2F_0$. Thus

$$B_n(z,\nu) = z^n \; {}_2F_0(-n,n+\nu+1;-1/z) = (n+1+\nu)_n \; {}_1F_1(-n,-2n-\nu;z) \quad . \tag{2.3}$$

Also,

$$P_n(z,\nu) = \frac{(-)^{n+1}e^{-z}}{z^\nu(\nu+1)_n}\int_0^z (z-t)^n t^{n+\nu} e^t dt$$

$$= \frac{(-)^{n+1} n! e^{-z} z^{2n+1}}{(\nu+1)_{2n+1}} \; {}_1F_1(n+\nu+1;2n+\nu+2;z) \quad , \tag{2.4}$$

$$R_n(z,\nu) = \frac{(-)^{n+1} n!\Gamma(\nu+1)\Gamma(n+\nu+1) z^{2n+1} \; {}_1F_1(n+1;2n+\nu+2;-z)}{\Gamma(2n+\nu+1)\Gamma(2n+\nu+2) \; {}_1F_1(-n;-2n-\nu;z)} \quad . \tag{2.5}$$

Both $A_n(z,\nu)$ and $B_n(z,\nu)$ satisfy the same recurrence equation

$$\frac{(n+\nu+1)}{(2n+\nu+1)(2n+\nu+2)} A_{n+1}(z,\nu) = \left[1 + \frac{\nu z}{(2n+\nu)(2n+\nu+2)}\right] A_n(z,\nu)$$

$$+ \frac{nz^2}{(2n+\nu)(2n+\nu+1)} A_{n-1}(z,\nu) \quad . \tag{2.6}$$

We list some other useful relations:

$$(2n+\nu)DB_n(z,\nu) - nB_n(z,\nu) + nB_{n-1}(z,\nu) = 0 \quad , \tag{2.7}$$

$$(2n+\nu)DA_n(z,\nu)+(n+\nu)A_n(z,\nu)+nzA_{n-1}(z,\nu)+z^{-1}\nu(2n+\nu)\left[A_n(z,\nu)-B_n(z,\nu)\right]= 0 \ , \tag{2.8}$$

and

$$\left[zD^2-(z+2n+\nu)D+n\right]B_n(z,\nu) = 0 \ , \ D = \frac{d}{dz} \quad . \tag{2.9}$$

We now develop a useful estimate of the remainder $R_n(z,\nu)$. Consider

$$y = e^{-z/2}{}_1F_1(a,c;z) \ , \ zy'' + cy' + (K-z/4)y = 0 \ , \ K = c/2 - a \quad . \tag{2.10}$$

Assume

$$y = \sum_{k=0}^{\infty} \frac{a_k(z)}{u^k} \ , \ a_o(z) = 1 \ , \ u = \tfrac{1}{2}(c-1) \quad . \tag{2.11}$$

Then the $a_k's$ can be generated by the expression

$$a_{k+1}(z) = -\tfrac{1}{2}z\, a_k'(z) + \tfrac{1}{2}\int_0^z (t/4-K)a_k(t)dt \quad . \tag{2.12}$$

Thus, for example,

$$a_1(z) = z^2/16 - Kz/2 \ , \ a_2(z) = z^4/512 - Kz^3/32 + z^2(2K^2-1)/16 + Kz/4 \ , \text{ and}$$

$$a_3(z) = \frac{z^6}{24576} - \frac{Kz^5}{1024} + \frac{z^4(4K^2-3)}{512} - \frac{K(4K^2-13)z^3}{192} - \frac{z^2(3K^2-1)}{16} - \frac{Kz}{8} \ . \tag{2.13}$$

Clearly the series (2.11) is absolutely convergent for all z as it can be found by multiplication of the series ${}_1F_1(a;c;z)$ by the series for $e^{-z/2}$. However, if z is fixed, c is large and $K \ll c$, then (2.11) is a useful representation of y for large c. The expressions (2.6)-(2.8) can also be deduced from some general results of Olver, see the discussion in [8, p. 76].

Apply (2.10)-(2.13) to the confluent functions in (2.5). Then, with the aid of the duplication formula for gamma functions, we get

$$R_n(z,\nu) = \frac{(-)^{n+1} n!\pi\Gamma(\nu+1)\Gamma(n+\nu+1)z^{2n+1}e^{-z} \sum_{k=0}^{\infty} \frac{a_k(z)}{u^k}}{2^{4n+2\nu}(2n+\nu+1)\left[\Gamma\left(n+\frac{\nu+1}{2}\right)\Gamma\left(n+\frac{\nu}{2}+1\right)\right]^2 \sum_{k=0}^{\infty} \frac{(-)^k a_k(z)}{u^k}} \ , \tag{2.14}$$

or

$$R_n(z,\nu) = \frac{(-)^{n+1}\pi n!\Gamma(\nu+1)\Gamma(n+\nu+1)z^{2n+1}e^{-z}}{2^{4n+2\nu}(2n+\nu+1)\left[\Gamma\left(n+\frac{\nu+1}{2}\right)\Gamma\left(n+\frac{\nu}{2}+1\right)\right]^2} e^{\frac{2a_1(z)}{u}}$$

$$\times\left\{1 + O(1/n^3)\right\} \ , \tag{2.15}$$

where $u = n + \frac{1}{2}(\nu+1)$, $K = -\nu/2$ and the a_k's are given by (2.12). It follows that for z and ν fixed,

$$\lim_{n\to\infty} R_n(z,\nu) = 0 \ . \tag{2.16}$$

Since $\Gamma(n+a)/\Gamma(n+b) = n^{a-b}[1+O(1/n)]$, it is convenient to write

$$R_n(z,\nu) = \frac{(-)^{n+1}\pi\Gamma(\nu+1)z^{2n+1}e^{-z}}{2^{4n+2\nu+1}n^{\nu}(n!)^2}\left\{1+O(1/n)\right\} . \tag{2.17}$$

Concerning the use of (2.1), it is important to know the nature of $z_{n,i}^{(\nu)}$, $i = 1,2,\ldots,n$, the zeros of $B_n(z,\nu)$. It is clear from (2.9) that the zeros are simple. If $\nu = 0$, $B_n(z,\nu)$ is essentially the modified Bessel function of half-odd order (see (4.2)). In this instance the zeros can be deduced from the work of Olver [9]. See also Grosswald [12]. The zeros lie in the left half plane and are always complex except when n is odd in which case there is a single real root. If $z_n^{(\nu)}$ is the magnitude of the smallest root(s) of $B_n(z,\nu)$, then from the work of Olver [9] $z_n^{(0)} \sim 1.32548n$. Here $2(t_0^2-1)^{\frac{1}{2}} = 1.32548$ where t_0 is the real zero of $t = \coth t$. Thomson [10] and Kublanovskaia and Smirnova [11] have tabulated $\frac{1}{2}z_{1,0}$, the former for $n = 1(1)9$ to 4d , and the latter for $n = 1(1)21$ to 5d . Salzer [13] has tabulated $z_{n,i}^{(-1)}$ for $n = 1(1)16$ to 15d . We have prepared some tables of $z_{n,i}^{(\nu)}$ for $\nu = -3/2, 1/2, 1$. For all the ν values mentioned above, the roots lie in the left half plane. If n is odd, there is a single real root, otherwise the roots are complex. On the basis of the data, we conjecture that $z_n^{(\nu)} \sim 1.32548n + \nu + 1 - 1/\pi$. An alternative conjecture is that $z_{2n}^{(\nu)} \sim \frac{1}{2}\pi^{\frac{1}{2}}(3n+\nu+1)$, $z_{2n+1}^{(\nu)} \sim 3/2\ n\pi^{\frac{1}{2}} + \nu + 2$. Thus as n increases, the magnitude of the smallest zero(s) of $B_n(z,\nu)$ increases linearly with n . We see from (2.17) that to achieve high accuracy, n must be considerably larger than z , and so the value of $z_n^{(\nu)}$ is not critical.

Another rational approximation for $\gamma(ze^{i\pi},\nu)$ is

$$\nu z^{-\nu}e^{-z-i\nu\pi}\gamma(ze^{i\pi},\nu) = S_n(z,\nu) + T_n(z,\nu) \quad ,$$

$$S_n(z,\nu) = \frac{C_n(z,\nu)}{D_n(z,\nu)} , \; T_n(z,\nu) = \frac{Q_n(z,\nu)}{D_n(z,\nu)} \quad , \tag{2.18}$$

where

$$C_n(z,\nu) = -\frac{\nu}{z}\sum_{k=0}^{n-1} \frac{(-n)_{k+1}(n+\nu)_{k+1}}{(\nu)_{k+1}(k+1)!} z^n {}_3F_1\left(\begin{matrix}-n+1+k, n+\nu+1+k, 1\\ k+2\end{matrix}\middle| -\frac{1}{z}\right) , \quad (2.19)$$

and $D_n(z,\nu)$ is z^n times the ${}_3F_1$ in (2.19) with $k+1$ set equal to zero. Thus $D_n(z,\nu)$ is $B_n(z,\nu)$ if in the latter we replace ν by $\nu-1$. Also

$$Q_n(z,\nu) = \frac{(-)^n \nu z^{-\nu} e^{-z}}{(\nu)_n} \int_0^z (z-t)^n t^{n+\nu-1} e^t dt$$

$$= \frac{(-)^n n! e^{-z} z^{2n}}{(\nu+1)_{2n}} {}_1F_1(n+\nu;2n+\nu+1;z) \quad . \quad (2.20)$$

Both $C_n(z,\nu)$ and $D_n(z,\nu)$ satisfy (2.6) if ν is replaced by $\nu-1$. Also, $D_n(z,\nu)$ satisfies (2.7) and (2.9) with ν replaced by $\nu-1$. The relation analogous to (2.8) reads

$$(2n+\nu-1)DC_n(z,\nu) + (n+\nu-1)C_n(z,\nu) + nzC_{n-1}(z,\nu)$$

$$+ z^{-1}\nu(2n+\nu-1)\left[C_n(z,\nu)-D_n(z,\nu)\right] = 0 \quad . \quad (2.21)$$

After the manner of deriving (2.15) and (2.17) we have

$$T_n(z,\nu) = \frac{(-)^n \pi n! \Gamma(\nu+1)\Gamma(n+\nu) z^{2n} e^{-z} e^{2a_1^*(z)/u^*}}{2^{4n+2\nu-2}(2n+\nu)\left[\Gamma\left(n+\frac{\nu}{2}\right)\Gamma\left(n+\frac{\nu+1}{2}\right)\right]^2} \left\{1+O(1/n^3)\right\}$$

$$= \frac{(-)^n \pi\Gamma(\nu+1) z^{2n} e^{-z} e^{2a_1^*(z)/u^*}}{2^{4n+2\nu-1} n^{\nu-1}(n!)^2} \left\{1+O(1/n)\right\} \quad , \quad (2.22)$$

where $u^* = n+\nu/2$, $a_1^*(z) = z(z+4\nu-4)/16$, whence for z and ν fixed, $\lim_{n\to\infty} T_n(z,\nu) = 0$.

If z and ν are positive and fixed, then for a given n, it is clear that $R_n(z,\nu)$ and $T_n(z,\nu)$ are of opposite sign. This leads to the inequalities

$$\frac{C_n(z,\nu)}{D_n(z,\nu)} \gtrless \nu z^{-\nu} e^{-z} \int_0^z t^{\nu-1} e^t dt \gtrless \frac{A_n(z,\nu)}{B_n(z,\nu)}, \tag{2.23}$$

where $>$ or $<$ sign is chosen according as n is odd or even, respectively. For example,

$$\frac{(\nu+1)(\nu+2)-z}{(\nu+1)(\nu+2)+z} < \nu z^{-\nu} e^{-z} \int_0^z t^{\nu-1} e^t dt < \frac{\nu+1}{\nu+1+z}, \ z > 0, \ \nu > 0, \tag{2.24}$$

$$\frac{2-z}{2+z} < e^{-z} < \frac{1}{z+1}, \ z > 0, \tag{2.25}$$

$$\frac{15-4z^2}{15+6z^2} < z^{-1} e^{-z^2} \int_0^z e^{t^2} dt < \frac{3}{2z^2+3}, \ z > 0 . \tag{2.26}$$

In the last three equations, equality prevails if $z \to 0$.

At this juncture, we present a summary of the material in the remaining sections of the report. If $\nu = \frac{1}{2}$, (2.1) yields approximations for the error function and related integrals which is the subject of Section III. Approximations for the exponential function and the circular functions which come from (2.1) when $\nu = 0$ are discussed in Section IV. Tables of the polynominals in the approximations of the functions discussed in Sections III and IV, and related data, are found in Appendix A. If $\nu = 0$, (1.3) is the exponential integral and in [1], we developed rational approximations for this function based on its asymptotic representation. Though $\Gamma(z,0)$ is defined by (1.3), clearly $\gamma(z,0)$ is not. However, we can give an ascending series representation for a function closely related to $\Gamma(z,0)$. We show in Section V how rational approximations to this related function may be derived, and tables of the polynomials in these approximations and related data are given in Appendix B. FORTRAN codes for computing rational approximations to the incomplete gamma function and its special cases are presented in Appendices C, D and E.

III. ERROR FUNCTION AND RELATED INTEGRALS

We list below formulae useful for the approximation of the error functions and the Fresnel integrals. These are based on (2.1) for $\nu = \frac{1}{2}$. It is clear from (2.1) and (2.18) that $V_n(z,\frac{1}{2})$ and $R_n(z,\frac{1}{2})$ can be replaced by $S_n(z,\frac{1}{2})$ and $T_n(z,\frac{1}{2})$, respectively.

$$\gamma(z,\tfrac{1}{2}) = \int_0^z t^{-\frac{1}{2}}e^{-t}dt = 2z^{\frac{1}{2}}e^{-z}\left\{V_n(ze^{-i\pi},\tfrac{1}{2}) + R_n(ze^{-i\pi},\tfrac{1}{2})\right\} . \tag{3.1}$$

$$\mathrm{Erf}(z) = \tfrac{1}{2}\gamma(z^2,\tfrac{1}{2}) = \int_0^z e^{-t^2}dt = (\pi/4)^{\frac{1}{2}} - \mathrm{Erfc}(z) , \tag{3.2}$$

$$\mathrm{Erfc}(z) = \int_z^\infty e^{-t^2}dt , \tag{3.3}$$

$$\mathrm{Erf}(z) = \int_0^z e^{-t^2}dt = ze^{-z^2}\left\{V_n(z^2e^{-i\pi},\tfrac{1}{2})+R_n(z^2e^{-i\pi},\tfrac{1}{2})\right\} , \tag{3.4}$$

$$\mathrm{Erfi}(z) = \int_0^z e^{t^2}dt = -\,i\mathrm{Erf}(iz) , \tag{3.5}$$

$$\gamma(ze^{\frac{i3\pi}{2}},\tfrac{1}{2}) = e^{\frac{i3\pi}{4}}\int_0^z t^{-\frac{1}{2}}e^{it}dt = 2z^{\frac{1}{2}}e^{iz+i3\pi/4}\left\{V_n(ze^{\frac{i\pi}{2}},\tfrac{1}{2})+R_n(ze^{\frac{i\pi}{2}},\tfrac{1}{2})\right\} , \tag{3.6}$$

$$C(z) + iS(z) = (2\pi)^{-\frac{1}{2}}\int_0^z t^{-\frac{1}{2}}e^{it}dt . \tag{3.7}$$

The exact coefficients of polynomials simply related to $A_n(z,\frac{1}{2})$, $B_n(z,\frac{1}{2})$, $C_n(z,\frac{1}{2})$ and $D_n(z,\frac{1}{2})$ are given in Appendix A for $n = 0(1)10$. For $n = 2(1)10$, we also record values of $R_n^*(z,\frac{1}{2}) = |2z^{\frac{1}{2}}e^zR_n(z,\frac{1}{2})|$, see (2.15), for $z = re^{i\theta}$, $r = 1(1)10$, $\theta = 0, \pi/2, \pi$. Error tables for other values of ν and for $T_n(z,\nu)$ are easily derived from the latter table. See the discussion in Appendix A. See Appendix C for FORTRAN codes.

To illustrate the utility of (2.15), we present the numerics below. The tables give $e^{-i\pi/2}\gamma(ze^{i\pi},\frac{1}{2})$, its rational approximation (2.1), the exact error, and the approximate error according to (2.15). The data are developed for $z = 2e^{i\theta}$, $n = 4$.

θ	$\int_0^z t^{-\frac{1}{2}}e^t dt$ Exact	$2z^{\frac{1}{2}}e^z V_n(z,\frac{1}{2})$
0	6.6876855	6.6877002
$\pi/2$	$3.3756999 \times 10^{-1} + 2.3328174i$	$3.3756168 \times 10^{-1} + 2.3328241i$
π	$1.0324115 \times 10^{-8} + 1.6918067i$	$1.0317960 \times 10^{-8} + 1.6917947i$

θ	Exact Error	$2z^{\frac{1}{2}}e^z R_n(z,\frac{1}{2})$ Approximate
0	-1.47×10^{-5}	-1.47×10^{-5}
$\pi/2$	$8.31 \times 10^{-6} - 6.70i \times 10^{-6}$	$8.35 \times 10^{-6} - 6.74i \times 10^{-6}$
π	$6.16 \times 10^{-12} + 1.20i \times 10^{-5}$	$1.05 \times 10^{-11} + 1.19i \times 10^{-5}$

IV. THE EXPONENTIAL FUNCTION

If $\nu \longrightarrow 0$, (2.1) becomes

$$e^{-z} = \frac{G_n(-z)}{G_n(z)} + R_n(z,0) \quad , \tag{4.1}$$

$$G_n(z) = z^n {}_2F_0(-n,n+1;-\tfrac{1}{2}) = (2/\pi)^{\frac{1}{2}} z^n K_{n+\frac{1}{2}}(z/2) \quad , \tag{4.2}$$

$$R_n(z,0) = \frac{(-)^{n+1}\pi I_{n+\frac{1}{2}}(z/2)}{e^z K_{n+\frac{1}{2}}(z/2)} = \frac{2(-)^{n+1}(z/4)^{2n+1}e^{-z+z^2/(8n+4)}}{n\left[(\frac{1}{2})_n\right]^2}\left\{1+O(1/n^3)\right\}$$

$$= \frac{(-)^{n+1}\pi z^{2n+1}e^{-z}}{2^{4n+1}(n!)^2}\left\{1+O(1/n)\right\} \quad . \tag{4.3}$$

Here $I_n(z)$ and $K_n(z)$ are the familiar notations for the modified Bessel functions.

It is of interest to show how (4.1) can be used to compute the exponential and circular functions in an efficient manner. Let

$$G_n(z) = M_n(z^2) + zN_n(z^2) \tag{4.4}$$

where $M_n(z^2)$ and $N_n(z^2)$ are even polynomials in z. Clearly

$$e_n^{-z} = \frac{G_n(-z)}{G_n(z)} = \frac{M_n(z^2)-zN_n(z^2)}{M_n(z^2)+zN_n(z^2)} \quad , \tag{4.5}$$

and one readily verifies that $G_n(z)$, $M_n(z^2)$ and $N_n(z^2)$ all satisfy the recurrence formula

$$G_{n+1}(z) = 2(2n+1)G_n(z) + z^2G_{n-1}(z) \quad . \tag{4.6}$$

Let the approximation to the circular functions be denoted by

$$e_n^{-iz} = \cos_n z - i \sin_n z \quad . \tag{4.7}$$

Then

$$\cos_n z = \frac{U_n(z^2)}{W_n(z^2)}, \quad \sin_n z = \frac{zV_n(z^2)}{W_n(z^2)} \quad ,$$

$$U_n(z^2) = \left[M_n(-z^2)\right]^2 - z^2\left[N_n(-z^2)\right]^2 , \quad V_n(z^2) = 2M_n(-z^2)N_n(-z^2) \quad ,$$

$$W_n(z^2) = \left[M_n(-z^2)\right]^2 + z^2\left[N_n(-z^2)\right]^2 \quad . \tag{4.8}$$

It can be shown that* $U_n(z^2)$, $V_n(z^2)$ and $W_n(z^2)$ all satisfy the recurrence formula

$$(2n-1)U_{n+1}(z^2) = \left[-z^2+4(4n^2-1)\right]\left[(2n+1)U_n(z^2)-z^2(2n-1)U_{n-1}(z^2)\right]$$

$$+ z^6(2n+1)U_{n-2}(z^2) \quad . \tag{4.9}$$

The exact coefficients of the polynomials $G_n(z)$, $U_n(z^2)$, $V_n(z^2)$ and $W_n(z^2)$ are given in Appendix A for $n = 1(1)10$. See also the introduction to Table A.III in Appendix A for the evaluation of $R_n(z,0)$. Appendix D gives FORTRAN codes for the computation of e_n^{-z}, $\cos_n z$, $\sin_n z$ and $\tan_n z$ for z real only. We conclude this section with an example to indicate the effectiveness of (4.1)-(4.3).

If $n = 4$ and $z = 1$, (4.5) gives the value $e_4^{-1} = 0.36787\ 94561$. The error is -1.5×10^{-8} whereas the last of (4.3) gives the value -1.6×10^{-8}. For $n = 4$ and $z = i$, (4.5) yields $e_4^{-i} = 0.54030\ 23380 + 0.8414709642i$. The true error is $-3.2 \times 10^{-8} - i2.1 \times 10^{-8}$ as compared with the estimated error from the last of (4.3), $-3.4 \times 10^{-8} - i2.2 \times 10^{-8}$.

V. RATIONAL APPROXIMATIONS FOR $E(z) = z^{-1}\int_0^z t^{-1}(1-e^{-t})dt$

In this section we develop some rational approximations for $E(z)$. It is of interest to first list some functions related to $E(z)$. We have

$$E(z) = z^{-1}\int_0^z t^{-1}(1-e^{-t})dt = z^{-1}\left[-Ei(-z)+\gamma+\ln z\right], \quad -Ei(-z) = \Gamma(z,0), \tag{5.1}$$

where γ is Euler's constant.

* We are indebted to Mr. Jet Wimp for proof of this statement and other helpful suggestions.

$$\tfrac{1}{2}\left[E(ze^{i\pi/2})+E(ze^{-i\pi/2})\right] = z^{-1}\mathrm{Si}(z) = z^{-1}\int_0^z t^{-1}\sin t\,dt$$

$$= z^{-1}\left[\pi/2 + \mathrm{si}(z)\right],\quad \mathrm{si}(z) = \int_\infty^z t^{-1}\sin t\,dt, \tag{5.2}$$

$$2i\left[E(ze^{i\pi/2})-E(ze^{-i\pi/2})\right] = 4z^{-2}H(z) = 4z^{-2}\int_0^z t^{-1}(1-\cos t)dt$$

$$= 4z^{-2}\left[\mathrm{Ci}(z)-\gamma-\ln z\right],\quad \mathrm{Ci}(z) = \int_\infty^z t^{-1}\cos t\,dt\,. \tag{5.3}$$

Approximations for $E(z)$ can be derived on the basis of the results given in Section II. Clearly from (1.1) and (2.1),

$$zE(z) = \lim_{\nu\to 0}\int_0^z t^{\nu-1}(1-e^{-t})dt = \lim_{\nu\to 0}\left[\frac{z^\nu}{\nu} - \gamma(z,\nu)\right]$$

$$= \frac{\partial}{\partial\nu}\left\{z^\nu - e^{-z}z^\nu\left[V_n(ze^{-i\pi},\nu)+R_n(ze^{-i\pi},\nu)\right]\right\}_{\nu=0}$$

$$= \frac{-e^{-z}}{\left[B_n(ze^{-i\pi},0)\right]^2}\left\{B_n(ze^{-i\pi},0)\frac{\partial A_n(ze^{-i\pi},0)}{\partial\nu} - A_n(ze^{-i\pi},0)\frac{\partial B_n(ze^{-i\pi},0)}{\partial\nu}\right\}$$

$$- e^{-z}\frac{\partial R_n(ze^{-i\pi},0)}{\partial\nu}\quad . \tag{5.4}$$

This approach is not satisfactory since the numerator and denominator polynomials of the rational approximation are now each of degree $2n$.

It calls for remark that the rational approximations given in Section II are of the Padé type. Let

$$E(z) = \sum_{k=0}^{\infty} a_k z^k \tag{5.5}$$

be approximated by

$$E_{p,q}(z) = \frac{f_p(z)}{g_q(z)} \quad , \tag{5.6}$$

where $f_p(z)$ and $g_q(z)$ are polynomials in z of degree p and q, respectively. If

$$g_q(z)E(z) - f_p(z) = z^{p+q+1}h(z) \ , \ h(0) \neq 0 \quad , \tag{5.7}$$

then (5.6) is the Padé approximant of $E(z)$. Thus (2.1) is of the type (5.6) where $p = q = n$, and (2.18) is also of the same type with $q = p+1 = n$. The approximation (5.6) is called the main diagonal Padé approximation if $p = q = n$.

The numerator and denominator polynomials in the Padé approximants to transcendental functions are known in closed form, as for the functions in Section II, only in very few cases. However, the Padé approximant to a Taylor series expansion can always be found by solving systems of linear equations. Thus, if

$$f(z) = \sum_{k=0}^{p} f_k z^k \ , \ g(z) = \sum_{k=0}^{q} g_k z^k \ , \tag{5.8}$$

then

$$\sum_{j=0}^{r} a_j g_{r-j} = 0 \ , \ r = p+1, p+2, \ldots, p+q \quad ,$$

$$f_k = \sum_{j=0}^{k} a_j g_{k-j} \ , \ k = 0,1,\ldots p \quad ,$$

$$h(z) = \sum_{k=0}^{\infty} h_k z^k \ , \ h_k = \sum_{j=0}^{p+q+1+k} a_j g_{p+q+1+k-j} \quad . \tag{5.9}$$

The coefficients f_k, g_k and h_k must be determined anew for each choice of p and q.

As remarked previously, it is inconvenient to use (5.4) as a rational approximation for $E(z)$. To circumvent this difficulty, we have computed the coefficients of the main diagonal Padé approximants of the functions $E(z)$ for $n = 0(1)10$ and $z^{-1}Si(z)$ and $4z^{-2}H(z)$ for $n = 0(2)10$. These are tabulated in Appendix B.

We also include tables of the absolute values of the errors incurred by using the Padé approximation of $E(z)$ for $n = 2(1)10$, $r = 1(1)10$, and $\theta = 0, \pi/2, \pi$, where $z = re^{i\theta}$, and the Padé approximants of $Si(z)$ and $H(z)$ for z real, $z = 1(1)10$ and for $n = 4(2)10$. These tables may be used as a guide in selecting the order of approximation necessary to obtain a desired accuracy. For example, if six decimal accuracy is desired for $E(z)$ for $z = 2^{\frac{1}{2}}e^{i\pi/4}$, i.e., $|\text{error}| < 0.5 \times 10^{-6}$, interpolation of Table B.II indicates that a third order approximation should be sufficient. The third order approximation gives 0.76507 22371 - 0.15899 83867i and the true value is 0.76507 22539 - 0.15899 86256i. Thus $|\text{error}| < 2.41 \times 10^{-7}$.

Now

$$E(iz) = z^{-1}Si(z) - iz^{-1}H(z) \quad . \tag{5.10}$$

Select n. It is readily deduced from the error tables that the Padé values for $E(iz)$ are better than the values deduced from (5.10) by using the Padé approximants for $S(z)$ and $H(z)$ for z real. Thus, if both $Si(z)$ and $H(z)$ are needed, it is better to use the Pade for $E(iz)$. However, if only $H(z)$, say, is needed, it is more economical to use the Pade for $H(z)$.

An examination of the zeros of the denominators of the Padé approximations of $E(z)$, $Si(z)$ and $H(z)$ indicates that the magnitudes of the smallest zeros of the denominator increase linearly with n. Since n must be significantly larger than the argument to attain good accuracy, the location of these zeros is not important.

As noted above, the Padé approximants for $E(z)$ are not known in closed form, and it is necessary to solve systems of linear equations for each selection of p and q (see (5.8)). We now give another rational approximation of $E(z)$ which is of the form (5.8) with $p = q = n$. In over-all accuracy, it is somewhat inferior to the corresponding Padé approximant, see Tables B.II and B.VIII. However, it has the desirable advantage that the numerator and denominator polynomials can be computed by recurrence formulas. Following [14], it can be shown that

$$E(z) = {}_2F_2\left(\begin{matrix}1,1\\2,2\end{matrix}\middle| -z\right) = \varphi_n(z)/f_n(z) + \epsilon_n(z) \ , \quad \epsilon_n(z) = F_n(z)/f_n(z) \quad , \qquad (5.11)$$

$$\varphi_n(z) = \sum_{r=0}^{m} \frac{(-)^r \binom{n}{r} (n+1)_r}{(2)_r} \, {}_4F_2\left(\begin{matrix}-n+r, n+1+r, 1, 2+r\\1+r, 1+r\end{matrix}\middle| -1/z\right) \ , \qquad (5.12)$$

and $f_n(z)$ is the ${}_4F_2$ in (5.12) with $r = 0$ whence it becomes a ${}_3F_1$, see (5.13). The nature of $F_n(z)$ will not be discussed here in detail. Suffice it to remark, it will be shown elsewhere that for z fixed $\lim_{n\to\infty} F_n(z) = 0$.

Now

$$f_n(z) = {}_3F_1\left(\begin{matrix}-n, n+1, 2\\1\end{matrix}\middle| -1/z\right)$$

$$= \frac{(n+1)(2n)!}{n!z^n} \, {}_2F_2\left(\begin{matrix}-n, -n\\-2n, -n-1\end{matrix}\middle| z\right) \quad , \qquad (5.13)$$

where it is to be understood that in the ${}_2F_2$ only the first $(n+1)$ terms of the series are retained. Thus

$$f_n(z) = \frac{(n+1)(2n)!}{n!z^n} \exp\left\{\frac{nz}{2(n+1)}\right\}\left\{1 - \frac{z^2(n^2+2n-2)}{8(n+1)^2(2n-1)} + O(z^3/n^3)\right\} \qquad (5.14)$$

and so for z fixed,

$$\lim_{n\to\infty} \epsilon_n(z) = 0 \quad , \qquad (5.15)$$

whence the rational approximants converge. It will also be demonstrated elsewhere that both $\varphi_n(z)$ and $f_n(z)$ satisfy the recurrence formula

$$f_n(z) = (B_1z+A_1)f_{n-1}(z) + (B_2z+A_2)f_{n-2}(z) + A_3f_{n-3}(z) \quad ,$$

$$A_1 = - A_3 = \frac{(n-2)(2n-1)}{n(2n-3)} \, , \; A_2 = 1 \quad ,$$

$$B_1 = \frac{2(2n-1)(n+1)}{n} \, , \text{ and } \; B_2 = \frac{-2(2n-1)(n-3)}{n} \quad . \tag{5.16}$$

To illustrate the power of (5.14), take $z = \frac{1}{2}$ and $n = 4$. The true value (from (5.12)) is 1101 and the value using (5.14) is 1098.5.

From the preceding development and the last remark, it is obvious that all that is needed for a useful expression for the error $|\epsilon_n(z)|$ is an approximation of $F_n(z)$ (see (5.11)) for large n. This is not available at the present time. If z is real and positive, we can show that

$$|\epsilon_n(z)| \le \left| \frac{5(2z)^{\frac{1}{2}}}{f_n(z)} \right| \, . \tag{5.17}$$

This bound, however, is very conservative.

The coefficients of the polynomials $\varphi_n(z)$ and $f_n(z)$ are given in Appendix B for $n = 0(1)10$. Also presented are values of $|\epsilon_n(z)|$ for $n = 3(1)10$ and for $z = re^{i\theta}$ where $r = 1(1)10$, $\theta = 0, \pi/2, \pi$.

The approximation (5.11) has the same properties as the Padé approximation in that the magnitude of the smallest zeros of the denominator polynomials increase linearly with n, the order of approximation. Again since the order of approximation must be significantly larger than the argument, the location of zeros of the denominator polynomials is not critical.

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APPENDIX A

COEFFICIENTS FOR RATIONAL APPROXIMATIONS TO THE ERROR FUNCTION, EXPONENTIAL FUNCTION AND CIRCULAR FUNCTIONS

We first show how the exact coefficients in the approximations (2.1) and (2.18) can be generated when ν is rational. Suppose $\nu = p/q$, $q \neq 0$ and p,q are co-prime integers.

Let

$$\left.\begin{aligned} A_n^*(z,\nu) &= q^{2n} \frac{\Gamma(\nu+n+1)}{\Gamma(\nu+1)} A_n(z,\nu) = \sum_{k=0}^{n} a_{n,k} z^k \quad , \\ B_n^*(z,\nu) &= q^{2n} \frac{\Gamma(\nu+n+1)}{\Gamma(\nu+1)} B_n(z,\nu) = \sum_{k=0}^{n} b_{n,k} z^k \quad , \end{aligned}\right\} \tag{A.1}$$

where $A_n(z,\nu)$ and $B_n(z,\nu)$ are defined in (2.2) and (2.3), respectively. Note that

$$\frac{A_n^*(z,\nu)}{B_n^*(z,\nu)} = \frac{A_n(z,\nu)}{B_n(z,\nu)} \quad , \tag{A.2}$$

and the transformation (A.1) insures that the coefficients $a_{n,k}$ and $b_{n,k}$ are integers if p and q are co-prime integers. If $\nu = 0$, set $p = 0$ and $q = 1$. From (2.6) it is seen that

$$\Lambda_{n+1} = \frac{(2nq+p+q)}{(2nq+p)}\left[(2nq+p)(2nq+p+2q)+pqz\right]\Lambda_n + \frac{q^3nz^2(2nq+p+2q)(nq+p)}{(2nq+p)}\Lambda_{n-1} ,$$

$$\Lambda_n = A_n^*(z,\nu) \quad \text{or} \quad B_n^*(z,\nu) \quad , \tag{A.3}$$

and

$$\left.\begin{aligned} &A_o^*(z,\nu) = 1 \ , \ A_1^*(z,\nu) = (p+q)(p+2q) - q^2 z \quad , \\ &B_o^*(z,\nu) = 1 \ , \ B_1^*(z,\nu) = (p+q)(p+2q) + q(p+q)z \quad . \end{aligned}\right\} \quad (A.4)$$

Using (A.3) and (A.1), we get the recurrence formula

$$\left.\begin{aligned} \lambda_{n+1,k} &= (2nq+p+q)(2nq+p+2q)\lambda_{n,k} + \frac{pq(2nq+p+q)}{(2nq+p)}\lambda_{n,k-1} \\ &\quad + \frac{q^3 n(2nq+p+2q)(nq+p)}{(2nq+p)}\lambda_{n-1,k-2} \quad , \\ &\lambda_{n,k} = a_{n,k} \ \text{ or } \ b_{n,k} \ ; \ k = 0,1,2,\ldots,n \\ &\text{and } \ \lambda_{n,k} = 0 \ \text{ if } \ k < 0 \ \text{ or } \ k > n \quad . \end{aligned}\right\} \quad (A.5)$$

The initial values are

$$\left.\begin{aligned} &a_{o,o} = 1 \quad , \\ &a_{1,o} = (p+q)(p+2q) \ , \quad a_{1,1} = -q^2 \ , \\ &b_{o,o} = 1 \quad , \\ &b_{1,o} = (p+q)(p+2q) \ , \quad b_{1,1} = q(p+q) \quad . \end{aligned}\right\} \quad (A.6)$$

Using a similar argument we find the following relations.

$$
\left.
\begin{aligned}
C_n^*(z,\nu) &= \frac{q^{2n}\Gamma(\nu+n+1)}{\Gamma(\nu+1)} C_n(z,\nu) = \sum_{k=0}^{n} c_{n,k} z^k \quad , \\
D_n^*(z,\nu) &= \frac{q^{2n}\Gamma(\nu+n+1)}{\Gamma(\nu+1)} D_n(z,\nu) = \sum_{k=0}^{n} d_{n,k} z^k \quad ,
\end{aligned}
\right\} \qquad (A.7)
$$

where $C_n(z,\nu)$ and $D_n(z,\nu)$ are defined in (2.19).

$$
\left.
\begin{aligned}
\Omega_{n+1} &= \frac{(2nq+p)}{(2nq+p-q)}\Big[(2nq+p-q)(2nq+p+q) + q(p-q)z\Big]\Omega_n \\
&\quad + \frac{q^3 n z^2 (2nq+p+q)(p+nq-q)}{(2nq+p-q)}\Omega_{n-1} \, , \\
\Omega_n &= C_n^*(z,\nu) \text{ or } D_n^*(z,\nu) \quad ,
\end{aligned}
\right\} \qquad (A.8)
$$

$$
\left.
\begin{aligned}
&C_o^*(z,\nu) = 0 \; , \quad C_1^*(z,\nu) = (p+q) \quad , \\
&D_o^*(z,\nu) = 1/p \; , \quad D_1^*(z,\nu) = (p+q) + qz \quad ,
\end{aligned}
\right\} \qquad (A.9)
$$

$$
\left.
\begin{aligned}
\gamma_{n+1,k} &= (2nq+p)(2nq+p+q)\gamma_{n,k} + \frac{q(p-q)(2nq+p)}{(2nq+p-q)}\gamma_{n,k-1} \\
&\quad + \frac{q^3 n(2nq+p+q)(p+nq-q)}{(2nq+p-q)}\gamma_{n-1,k-2} \quad , \\
&\gamma_{n,k} = c_{n,k} \text{ or } d_{n,k} \; ; \; k = 0,1,\ldots,n \\
&\text{and } \gamma_{n,k} = 0 \text{ if } k < 0 \text{ or } k > n \quad ,
\end{aligned}
\right\} \qquad (A.10)
$$

$$\left.\begin{aligned}
&c_{o,o} = 0 \quad , \\
&c_{1,1} = 0 \ , \ c_{1,o} = (p+q) \quad , \\
&d_{o,o} = 1/p \quad , \\
&d_{1,o} = (p+q) \ , \ d_{1,1} = q \quad .
\end{aligned}\right\} \qquad (A.11)$$

Tables A.I and A.II give the coefficients of the polynomials in the rational approximations to the error function, see (2.1), (2.18), (3.2) and (3.3). Table A.III gives the error associated with the approximation (2.1) for $\nu = \frac{1}{2}$. The introduction to Table A.III shows how the table may be used to get $R_n(z,\nu)$ for other values of ν and $T_n(z,\nu)$. Table A.IV gives coefficients of the polynomials in the rational approximation to the exponential function, see (4.1). Tables A.V, A.VI and A.VII list the coefficients of $U_n(z^2)$, $W_n(z^2)$ and $V_n(z^2)$, respectively, see (4.7) and (4.8). These coefficients are pertinent to the evaluation of the circular functions.

Most of the tables in the Appendices were typed by the IBM 1620 computer directly on stencils, while the balance of the report was done on an ordinary typewriter. The typewriters have different type sizes. The computer has no lower case characters, etc., and so a slight variance in notation is introduced. For example, in Table A.I, AN*(Z,1/2) corresponds to $A_n(z,\frac{1}{2})$, etc. Also in Table A.V, UN(Z**2) corresponds to $U_n(z^2)$, etc.

TABLE A.I

TABLES OF THE COEFFICIENTS OF THE POLYNOMIALS
$A_n^*(z,\frac{1}{2})$ AND $B_n^*(z,\frac{1}{2})$

Note: For the definition of $A_n^*(z,\frac{1}{2})$ and $B_n^*(z,\frac{1}{2})$, see (2.1) and (A.1). The sequence of numbers given is for the highest power to the lowest power, respectively.

$$\text{e.g., } A_3^*(z,\tfrac{1}{2}) = -128z^3 + 1932z^2 - 9240z + 45045$$

	AN*(Z, 1/2)	BN*(Z, 1/2)
N = 00		
	1	1
N = 01		
	-4	6
	15	15
N = 02		
	32	60
	-210	420
	945	945
N = 03		
	-128	280
	1932	3780
	-9240	20790
	45045	45045

TABLE A.I (Continued)

	AN*(Z,1/2)	BN*(Z,1/2)
N = 04		
	2048	5040
	-43560	1 10880
	5 40540	10 81080
	-22 52250	54 05400
	114 86475	114 86475
N = 05		
	-8192	22176
	2 81424	7 20720
	-38 91888	108 10800
	453 33288	918 91800
	- 1745 94420	4364 86050
	9166 20705	9166 20705
N = 06		
	65536	1 92192
	-28 85792	86 48640
	699 73904	1837 83600
	-7914 94704	23279 25600
	89625 13560	1 83324 14100
	-3 27946 51890	8 43291 04860
	17 56856 35125	17 56856 35125
N = 07		
	-2 62144	8 23680
	161 40480	490 08960
	-4353 52320	13967 55360
	91454 22000	2 44432 18800
	-9 17867 80800	28 10970 16200
	102 39962 73300	210 82276 21500
	-361 41044 94000	948 70242 96750
	1965 16931 86125	1965 16931 86125

TABLE A.I (Concluded)

N = 08

AN*(Z,1/2)	BN*(Z,1/2)
83 88608	280 05120
-6277 25952	21283 89120
2 51360 15040	7 82183 00160
-53 38931 08320	179 90209 03680
1038 33795 78000	2810 97016 20000
-9618 68096 04600	30358 47774 96000
1 06381 16578 08900	2 20098 96368 46000
-3 65521 49326 19250	9 74723 98203 18000
20 10368 21294 05875	20 10368 21294 05875

N = 09

AN*(Z,1/2)	BN*(Z,1/2)
-335 54432	1182 43840
32467 92960	1 11740 42880
-14 34070 38720	51 40059 72480
473 05494 72000	1499 18408 64000
-8681 49968 40000	30358 47774 96000
1 61017 72510 82400	4 40197 92736 92000
-14 07934 64071 26000	45 48711 91614 84000
154 87281 04783 86000	321 65891 40704 94000
-521 20657 37253 37500	1407 25774 90584 11250
2892 69648 41756 23125	2892 69648 41756 23125

N = 10

AN*(Z,1/2)	BN*(Z,1/2)
2684 35456	9932 48256
-3 05076 71040	11 42235 49400
182 46049 11200	642 50746 50000
-6185 20520 21760	23130 26876 16000
1 82600 62172 35200	58693 05698 25600
-30 32474 61076 56000	109 16908 59875 61600
544 83460 50675 52800	1501 07493 23289 72000
-4560 40860 39327 81600	15010 74932 32897 20000
49985 79524 65547 67600	1 04137 07343 03224 32500
-1 65462 23889 48456 42750	4 51260 65153 13972 07500
9 25084 33563 93642 75375	9 25084 33563 93642 75375

TABLE A.II

TABLES OF THE COEFFICIENTS OF THE POLYNOMIALS
$C_n^*(z,\frac{1}{2})$ AND $D_n^*(z,\frac{1}{2})$

Note: For the definition of $C_n^*(z,\frac{1}{2})$ and $D_n^*(z,\frac{1}{2})$, see (2.18) and (A.7). The sequence of numbers given is for the highest power to the lowest power, respectively.

e.g., $C_3^*(z,\frac{1}{2}) = 84z^2 - 420z + 3465$

	CN*(Z,1/2)	DN*(Z,1/2)
N = 00		
	0	1
N = 01		
	0	2
	3	3
N = 02		
	0	12
	-10	60
	105	105
N = 03		
	0	40
	84	420
	-420	1890
	3465	3465

TABLE A.II (Continued)

N = 04

CN*(Z,1/2)	DN*(Z,1/2)
0	560
-744	10080
23100	83160
-90090	3 60360
6 75675	6 75675

N = 05

CN*(Z,1/2)	DN*(Z,1/2)
0	2016
5104	55440
-82368	7 20720
17 29728	54 05400
-61 26120	229 72950
436 48605	436 48605

N = 06

CN*(Z,1/2)	DN*(Z,1/2)
0	14784
-25376	5 76576
15 53552	108 10800
-171 53136	1225 22400
3026 30328	8729 72100
-10184 67450	36664 82820
70274 25405	70274 25405

N = 07

CN*(Z,1/2)	DN*(Z,1/2)
0	54912
1 58528	28 82880
-53 60576	735 13440
2061 87696	11639 62800
-19288 52640	1 22216 09400
3 07868 16060	8 43291 04860
-10 03917 91500	35 13712 70250
67 76445 92625	67 76445 92625

TABLE A.II (Concluded)

N = 08

CN*(Z, 1/2)	DN*(Z, 1/2)
0	16 47360
-33 70624	1120 20480
3395 37600	37246 80960
-73102 91040	7 82183 00160
22 56425 02800	112 43880 64800
-192 17857 23000	1124 38806 48000
2873 21307 27300	7589 61943 74000
-9170 79015 35250	31442 70909 78000
60920 24887 69875	60920 24887 69875

N = 09

CN*(Z, 1/2)	DN*(Z, 1/2)
0	62 23360
198 88896	5320 97280
-11584 58880	2 23480 85760
7 01345 14560	59 96736 34560
-122 38237 44000	1124 38806 48000
3311 97300 72000	15179 23887 48000
-26551 62101 59200	1 46732 64245 64000
3 79059 32634 57000	9 74723 98203 18000
-11 91329 31137 22000	40 20736 42588 11750
78 18098 60588 00625	78 18098 60588 00625

N = 10

CN*(Z, 1/2)	DN*(Z, 1/2)
0	472 97536
-1105 68960	49662 41280
1 65206 97600	25 70029 86240
-58 55525 91360	856 67662 80000
2759 72254 27200	20238 98516 64000
-42544 70268 04800	3 52158 34189 53600
10 54145 93612 32800	45 48711 91614 84000
-81 01039 31733 09600	428 87855 20939 92000
1117 46689 40671 23600	2814 51549 81168 22500
-3471 23578 10107 47750	11570 78593 67024 92500
22563 03257 65698 60375	22563 03257 65698 60375

TABLE A.III

TABLES OF THE ERROR FOR THE ERROR FUNCTION

Here we give the values of $R_n^*(z,\frac{1}{2}) = |2z^{\frac{1}{2}}e^z R_n(z,\frac{1}{2})|$, see (2.15), for $n = 2(1)10$, $r = 1(1)10$ and $\theta = 0, \pi/2, \pi$ where $z = re^{i\theta}$.

It is pertinent to point out that $R_n^*(z,\nu)$, see (2.15), and $T_n^*(z,\nu) = |\nu^{-1}z^\nu e^z T_n(z,\nu)|$, see (2.22), can both be obtained from $R_n^*(z,\frac{1}{2})$ since

$$R_n(z,\nu) = 2^{2-\nu}\Gamma(\nu+1)n^{\frac{1}{2}-\nu}\pi^{-\frac{1}{2}}R_n(z,\tfrac{1}{2})\left\{1+O(1/n)\right\} ,$$

and

$$T_n(z,\nu) = -2^{4-2\nu}\Gamma(\nu+1)n^{3/2-\nu}z^{-1}\pi^{-\frac{1}{2}}R_n(z,\tfrac{1}{2})\left\{1+O(1/n)\right\} .$$

Hence, the tables given here essentially include the values of $R_n^*(z,\nu)$ and $T_n^*(z,\nu)$ for admissible ν fixed, but otherwise arbitrary and the values of n, r and θ mentioned above.

r \ θ	0	π/2	π
n = 2			
1	0.896 (-3)	0.747 (-3)	0.747 (-3)
2	0.509 (-1)	0.295 (-1)	0.354 (-1)
3	0.651 (	0.219	0.377
4	0.477 (1)	0.774	0.230 (1)
5	0.268 (2)	0.175 (1)	0.108 (2)
6	0.132 (3)	0.290 (1)	0.443 (2)
7	0.609 (3)	0.375 (1)	0.171 (3)
8	0.275 (4)	0.395 (1)	0.642 (3)
9	0.125 (5)	0.349 (1)	0.243 (4)
10	0.578 (5)	0.262 (1)	0.938 (4)

TABLE A.III (Continued)

r \ θ	0	π/2	π
n = 3			
1	0.521 (- 5)	0.456 (- 5)	0.456 (- 5)
2	0.111 (- 2)	0.746 (- 3)	0.853 (- 3)
3	0.294 (- 1)	0.132 (- 1)	0.197 (- 1)
4	0.344	0.905 (- 1)	0.202
5	0.264 (1)	0.358	0.136 (1)
6	0.160 (2)	0.973	0.719 (1)
7	0.838 (2)	0.200 (1)	0.330 (2)
8	0.402 (3)	0.331 (1)	0.138 (3)
9	0.183 (4)	0.454 (1)	0.552 (3)
10	0.813 (4)	0.531 (1)	0.214 (4)
n = 4			
1	0.178 (- 7)	0.160 (- 7)	0.160 (- 7)
2	0.147 (- 4)	0.107 (- 4)	0.119 (- 4)
3	0.833 (- 3)	0.443 (- 3)	0.608 (- 3)
4	0.162 (- 1)	0.567 (- 2)	0.107 (- 1)
5	0.181	0.373 (- 1)	0.108
6	0.144 (1)	0.158	0.765
7	0.923 (1)	0.484	0.442 (1)
8	0.513 (2)	0.116 (1)	0.221 (2)
9	0.259 (3)	0.227 (1)	0.100 (3)
10	0.123 (4)	0.375 (1)	0.428 (3)
n = 5			
1	0.401 (-10)	0.368 (-10)	0.368 (-10)
2	0.130 (- 6)	0.998 (- 7)	0.109 (- 6)
3	0.160 (- 4)	0.949 (- 5)	0.123 (- 4)
4	0.532 (- 3)	0.223 (- 3)	0.375 (- 3)
5	0.879 (- 2)	0.238 (- 2)	0.569 (- 2)
6	0.949 (- 1)	0.153 (- 1)	0.563 (- 1)
7	0.774	0.678 (- 1)	0.421
8	0.520 (1)	0.227	0.259 (1)
9	0.305 (2)	0.609	0.139 (2)
10	0.161 (3)	0.135 (1)	0.677 (2)

TABLE A.III (Continued)

r \ θ	0	π/2	π
n = 6			
1	0.639 (-13)	0.593 (-13)	0.593 (-13)
2	0.812 (- 9)	0.650 (- 9)	0.700 (- 9)
3	0.220 (- 6)	0.141 (- 6)	0.176 (- 6)
4	0.127 (- 4)	0.603 (- 5)	0.941 (- 5)
5	0.316 (- 3)	0.104 (- 3)	0.218 (- 3)
6	0.470 (- 2)	0.993 (- 3)	0.302 (- 2)
7	0.498 (- 1)	0.625 (- 2)	0.296 (- 1)
8	0.414	0.287 (- 1)	0.229
9	0.288 (1)	0.103	0.148 (1)
10	0.176 (2)	0.300	0.841 (1)
n = 7			
1	0.757 (-16)	0.710 (-16)	0.710 (-16)
2	0.380 (-11)	0.313 (-11)	0.334 (-11)
3	0.228 (- 8)	0.155 (- 8)	0.188 (- 8)
4	0.228 (- 6)	0.120 (- 6)	0.176 (- 6)
5	0.866 (- 5)	0.329 (- 5)	0.627 (- 5)
6	0.180 (- 3)	0.465 (- 4)	0.122 (- 3)
7	0.250 (- 2)	0.411 (- 3)	0.159 (- 2)
8	0.261 (- 1)	0.256 (- 2)	0.156 (- 1)
9	0.220	0.121 (- 1)	0.123
10	0.158 (1)	0.455 (- 1)	0.829
n = 8			
1	0.693 (-19)	0.655 (-19)	0.655 (-19)
2	0.138 (-13)	0.116 (-13)	0.123 (-13)
3	0.184 (-10)	0.131 (-10)	0.155 (-10)
4	0.322 (- 8)	0.182 (- 8)	0.256 (- 8)
5	0.187 (-6)	0.793 (- 7)	0.140 (- 6)
6	0.547 (- 5)	0.165 (- 5)	0.388 (- 5)
7	0.101 (- 3)	0.203 (- 4)	0.674 (- 4)
8	0.133 (- 2)	0.170 (- 3)	0.840 (- 3)
9	0.138 (- 1)	0.104 (- 2)	0.817 (- 2)
10	0.117	0.503 (- 2)	0.659 (- 1)

TABLE A.III (Concluded)

θ / r	0	π/2	π
n = 9			
1	0.505 (-22)	0.480 (-22)	0.480 (-22)
2	0.400 (-16)	0.343 (-16)	0.361 (-16)
3	0.119 (-12)	0.872 (-13)	0.102 (-12)
4	0.364 (-10)	0.218 (-10)	0.296 (-10)
5	0.325 (- 8)	0.151 (- 8)	0.251 (- 8)
6	0.134 (- 6)	0.458 (- 7)	0.987 (- 7)
7	0.329 (- 5)	0.783 (- 6)	0.230 (- 5)
8	0.553 (- 4)	0.873 (- 5)	0.367 (- 4)
9	0.701 (- 3)	0.698 (- 4)	0.442 (- 3)
10	0.716 (- 2)	0.427 (- 3)	0.429 (- 2)
n = 10			
1	0.300 (-25)	0.286 (-25)	0.286 (-25)
2	0.943 (-19)	0.820 (-19)	0.860 (-19)
3	0.625 (-15)	0.473 (-15)	0.544 (-15)
4	0.337 (-12)	0.217 (-12)	0.280 (-12)
5	0.464 (-10)	0.231 (-10)	0.368 (-10)
6	0.272 (- 8)	0.102 (- 8)	0.206 (- 8)
7	0.891 (- 7)	0.242 (- 7)	0.643 (- 7)
8	0.192 (- 5)	0.359 (- 6)	0.132 (- 5)
9	0.301 (- 4)	0.371 (- 5)	0.198 (- 4)
10	0.370 (- 3)	0.287 (- 4)	0.232 (- 3)

TABLE A.IV

TABLE OF THE COEFFICIENTS OF THE POLYNOMIAL $G_n(z)$

Note: For the definition of $G_n(z)$, see (4.1). The sequence of numbers given is for the highest power to the lowest power, respectively.

$$\text{e.g.,}\quad G_4(z) = z^4 + 20z^3 + 180z^2 + 840z + 1680$$

GN(Z)

N = 00

1	

N = 01

1	2

N = 02

1	6
12	

N = 03

1	12
60	120

N = 04

1	20
180	840
1680	

TABLE A.IV (Continued)

GN(Z)

N = 05

1	30
420	3360
15120	30240

N = 06

1	42
840	10080
75600	3 32640
6 65280	

N = 07

1	56
1512	25200
2 77200	19 95840
86 48640	172 97280

N = 08

1	72
2520	55440
8 31600	86 48640
605 40480	2594 59200
5189 18400	

N = 09

1	90
3960	1 10880
21 62160	302 70240
3027 02400	20756 73600
88216 12800	1 76432 25600

TABLE A.IV (Concluded)

GN(Z)

N = 10

1	110
5940	2 05920
50 45040	908 10720
12108 09600	1 17621 50400
7 93945 15200	33 52212 86400
67 04425 72800	

TABLE A.V

TABLE OF THE COEFFICIENTS OF THE POLYNOMIAL $U_n(z^2)$

Note: For the definition of $U_n(z^2)$, see (4.8). The sequence of numbers given is for the lowest power to the highest power, respectively.

$$\text{e.g.,} \quad U_3(z\ ) = -z^6 + 264z^4 - 6480z^2 + 14400$$

UN(Z**2)

N = 00

.10000 00000 00000 00000 +01

N = 01

.40000 00000 00000 00000 +01 -.10000 00000 00000 00000 +01

N = 02

.14400 00000 00000 00000 +03 -.60000 00000 00000 00000 +02
.10000 00000 00000 00000 +01

N = 03

.14400 00000 00000 00000 +05 -.64800 00000 00000 00000 +04
.26400 00000 00000 00000 +03 -.10000 00000 00000 00000 +01

N = 04

.28224 00000 00000 00000 +07 -.13104 00000 00000 00000 +07
.69360 00000 00000 00000 +05 -.76000 00000 00000 00000 +03
.10000 00000 00000 00000 +01

TABLE A.V (Continued)

UN(Z**2)

N = 05

.91445 76000 00000 00000 +09	-.43182 72000 00000 00000 +09
.25804 80000 00000 00000 +08	-.40824 00000 00000 00000 +06
.17400 00000 00000 00000 +04	-.10000 00000 00000 00000 +01

N = 06

.44259 74784 00000 00000 +12	-.21123 97056 00000 00000 +12
.13539 05280 00000 00000 +11	-.25788 67200 00000 00000 +09
.17035 20000 00000 00000 +07	-.34440 00000 00000 00000 +04
.10000 00000 00000 00000 +01	

N = 07

.29919 58953 98400 00000 +15	-.14384 41804 80000 00000 +15
.96499 66233 60000 00000 +13	-.20552 09587 20000 00000 +12
.17141 24160 00000 00000 +10	-.56629 44000 00000 00000 +07
.61600 00000 00000 00000 +04	-.10000 00000 00000 00000 +01

N = 08

.26927 63058 58560 00000 +18	-.13015 02144 98304 00000 +18
.90161 53232 48640 00000 +16	-.20687 40850 17600 00000 +15
.19940 43744 00000 00000 +13	-.86313 42720 00000 00000 +10
.15996 96000 00000 00000 +08	-.10224 00000 00000 00000 +05
.10000 00000 00000 00000 +01	

N = 09

.31128 34095 72495 36000 +21	-.15106 40075 86652 16000 +21
.10717 19697 31706 88000 +20	-.25935 10574 05440 00000 +18
.27586 13803 54560 00000 +16	-.14176 33176 96000 00000 +14
.35472 72960 00000 00000 +11	-.39964 32000 00000 00000 +08
.16020 00000 00000 00000 +05	-.10000 00000 00000 00000 +01

TABLE A.V (Concluded)

UN(7**2)

N = 10

.44949	32434	22683	29984	+24	-.21883	22369	29464	23808	+24
.15812	89202	64694	57920	+23	-.39825	96563	64810	24000	+21
.45494	37982	88179	20000	+19	-.26326	17302	51520	00000	+17
.79982	79569	28000	00000	+14	-.12473	80992	00000	00000	+12
.90676	08000	00000	00000	+08	-.23980	00000	00000	00000	+05
.10000	00000	00000	00000	+01					

TABLE A.VI

TABLES OF THE COEFFICIENTS OF THE POLYNOMIAL $W_n(z^2)$

Note: For the definition of $W_n(z^2)$, see (4.8). The sequence of numbers given is for the lowest power to the highest power, respectively.

e.g., $W_3(z^2) = z^6 + 24z^4 + 720z^2 + 14400$

WN(Z**2)

N = 00

.10000 00000 00000 00000 +01

N = 01

.40000 00000 00000 00000 +01 .10000 00000 00000 00000 +01

N = 02

.14400 00000 00000 00000 +03 .12000 00000 00000 00000 +02
.10000 00000 00000 00000 +01

N = 03

.14400 00000 00000 00000 +05 .72000 00000 00000 00000 +03
.24000 00000 00000 00000 +02 .10000 00000 00000 00000 +01

N = 04

.28224 00000 00000 00000 +07 .10080 00000 00000 00000 +06
.21600 00000 00000 00000 +04 .40000 00000 00000 00000 +02
.10000 00000 00000 00000 +01

TABLE A.VI (Continued)

WN(Z**2)

N = 05

.91445 76000 00000 00000 +09	.25401 60000 00000 00000 +08
.40320 00000 00000 00000 +06	.50400 00000 00000 00000 +04
.60000 00000 00000 00000 +02	.10000 00000 00000 00000 +01

N = 06

.44259 74784 00000 00000 +12	.10059 03360 00000 00000 +11
.12700 80000 00000 00000 +09	.12096 00000 00000 00000 +07
.10080 00000 00000 00000 +05	.84000 00000 00000 00000 +02
.10000 00000 00000 00000 +01	

N = 07

.29918 58953 98400 00000 +15	.57537 67219 20000 00000 +13
.60354 20160 00000 00000 +11	.46569 60000 00000 00000 +09
.30240 00000 00000 00000 +07	.18144 00000 00000 00000 +05
.11200 00000 00000 00000 +03	.10000 00000 00000 00000 +01

N = 08

.26927 63058 58560 00000 +18	.44879 38430 97600 00000 +16
.40276 37053 44000 00000 +14	.26153 48736 00000 00000 +12
.13970 88000 00000 00000 +10	.66528 00000 00000 00000 +07
.30240 00000 00000 00000 +05	.14400 00000 00000 00000 +03
.10000 00000 00000 00000 +01	

N = 09

.31128 34095 72495 36000 +21	.45776 97199 59552 00000 +19
.35903 50744 78080 00000 +17	.20138 18526 72000 00000 +15
.91537 20576 00000 00000 +12	.36324 28800 00000 00000 +10
.13305 60000 00000 00000 +08	.47520 00000 00000 00000 +05
.18000 00000 00000 00000 +03	.10000 00000 00000 00000 +01

TABLE A.VI (Concluded)

WN(Z**2)

N = 10

.44949 32434 22683 29984 +24	.59143 84781 87741 18400 +22
.41199 27479 63596 80000 +20	.20345 32088 70912 00000 +18
.80552 74106 88000 00000 +15	.27461 16172 80000 00000 +13
.84756 67200 00000 00000 +10	.24710 40000 00000 00000 +08
.71280 00000 00000 00000 +05	.22000 00000 00000 00000 +03
.10000 00000 00000 00000 +01	

TABLE A.VII

TABLES OF THE COEFFICIENTS OF THE POLYNOMIAL $V_n(z^2)$

Note: For the definition of $V_n(z^2)$, see (4.8). The sequence of numbers given is for the lowest power to the highest power, respectively.

e.g., $V_3(z^2) = 24z^4 - 1680z^2 + 14400$

VN(Z**2)

N = 00

.00000 00000 00000 00000 00

N = 01

.40000 00000 00000 00000 01

N = 02

.14400 00000 00000 00000 03 -.12000 00000 00000 00000 02

N = 03

.14400 00000 00000 00000 +05 -.16800 00000 00000 00000 +04
.24000 00000 00000 00000 +02

N = 04

.28224 00000 00000 00000 +07 -.36960 00000 00000 00000 +06
.88800 00000 00000 00000 +04 -.40000 00000 00000 00000 +02

TABLE A.VII (Continued)

N = 05

.91445 76000 00000 00000 +09	-.12700 80000 00000 00000 +09
.37900 80000 00000 00000 +07	-.31920 00000 00000 00000 +05
.60000 00000 00000 00000 +02	

N = 06

.44259 74784 00000 00000 +12	-.63707 21280 00000 00000 +11
.21388 14720 00000 00000 +10	-.23950 08000 00000 00000 +08
.90720 00000 00000 00000 +05	-.84000 00000 00000 00000 +02

N = 07

.29919 58953 98400 00000 +15	-.44112 21534 72000 00000 +14
.15946 92126 72000 00000 +13	-.21009 54240 00000 00000 +11
.11124 28800 00000 00000 +09	-.21974 40000 00000 00000 +06
.11200 00000 00000 00000 +03	

N = 08

.26927 63058 58560 00000 +18	-.40391 44587 87840 00000 +17
.15362 55847 52640 00000 +16	-.22479 54508 80000 00000 +14
.14503 37011 20000 00000 +12	-.41646 52800 00000 00000 +09
.47376 00000 00000 00000 +06	-.14400 00000 00000 00000 +03

N = 09

.31128 34095 72495 36000 +21	-.47302 87106 24870 40000 +20
.18669 82387 28601 60000 +19	-.29397 64065 63840 00000 +17
.21608 80001 28000 00000 +15	-.77785 86816 00000 00000 +12
.13278 98880 00000 00000 +10	-.93456 00000 00000 00000 +06
.18000 00000 00000 00000 +03	

TABLE A.VII (Concluded)

N = 10

.44949	32434	22683	29984	+24		-.69001	15578	85698	04800	+23
.28012	45506	33915	18720	+22		-.46561	72008	73144	32000	+20
.37541	77850	14272	00000	+18		-.15728	18837	99040	00000	+16
.34464	83040	00000	00000	+13		-.37378	59840	00000	00000	+10
.17186	40000	00000	00000	+07		-.22000	00000	00000	00000	+03

APPENDIX B

COEFFICIENTS OF RATIONAL APPROXIMATIONS TO $E(z)$ AND RELATED INTEGRALS

Table B.I gives the coefficients of the polynomials of the main diagonal Padé approximants of $E(z)$, see (5.1). Displayed in Table B.II is the absolute value of the errors associated with the approximations listed in B.I. Similar coefficients for the main diagonal Padé approximants to $z^{-1}Si(z)$, see (5.2), are given for n even in Table B.III. Table B.IV gives the error associated with the approximations listed in B.III. Table B.V lists the coefficients of the polynomials of the main diagonal Padé approximants to $4z^{-2}H(z)$, see (5.3) , for even n . Table B.VI lists the errors associated with the approximants in B.V. Tables B.VII and B.VIII give the coefficients which define the approximations (5.11) to $E(z)$, and the corresponding error tables, respectively.

TABLE B.I

COEFFICIENTS OF PADÉ APPROXIMATION TO E(z)

Note: For the definition of E(z) and its Padé approximants, see Section V. The coefficients are given for $n = 1(1)10$. The expression attached to the numbers on the right indicates the power of 10 by which the number is multiplied.

$$\text{e.g.,}\quad E_2(z) = \frac{1 + 0.086z + 0.04555\ldots z^2}{1 + 0.336Z + 0.0333\ldots z^2}$$

N = 01

NUMERATOR	DENOMINATOR
.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
-.27777 77777 77777 77778 -01	.22222 22222 22222 22222 +00

N = 02

NUMERATOR	DENOMINATOR
.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
.86000 00000 00000 00000 -01	.33600 00000 00000 00000 +00
.45555 55555 55555 55556 -02	.33000 00000 00000 00000 -01

N = 03

NUMERATOR	DENOMINATOR
.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
.11548 60120 33992 92848 +00	.36548 60120 33992 92848 +00
.14987 35259 04789 47405 -01	.50803 30004 34216 23969 -01
-.85008 16740 06988 81376 -04	.27277 05063 78843 33065 -02

N = 04

NUMERATOR	DENOMINATOR
.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
.15183 03075 00550 75001 +00	.40183 03075 00550 75001 +00
.22174 10124 06340 46892 -01	.67076 12256 02161 78840 -01
.71679 28213 71348 66118 -03	.55785 84155 83924 05926 -02
.10529 22995 44767 84505 -04	.19778 97186 99680 64309 -03

TABLE B.I (Continued)

N	NUMERATOR	DENOMINATOR
N = 05		
	.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
	.16204 16009 62342 70975 +00	.41204 16009 62342 70975 +00
	.26595 51845 22923 74506 -01	.74050 36313 73224 96387 -01
	.12747 83272 48448 67889 -02	.73128 40670 01829 36777 -02
	.60877 80145 04733 48711 -04	.40061 22491 28200 19535 -03
	-.13654 42296 45193 12376 -06	.98510 21164 96373 17832 -05
N = 06		
	.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
	.17964 76037 82301 29344 +00	.42964 76037 82301 29344 +00
	.30906 39014 59549 31654 -01	.82762 73553 59746 99458 -01
	.19643 87594 46735 53361 -02	.92024 26823 88873 61206 -02
	.11565 07429 94841 78517 -03	.62715 69030 34069 87432 -03
	.19803 30657 69150 50495 -05	.25037 59209 97727 45364 -04
	.12947 26971 80815 10906 -07	.46180 79516 66610 03704 -06
N = 07		
	.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
	.18480 11227 26475 81023 +00	.43480 11227 26475 81023 +00
	.33261 21378 35356 62215 -01	.86405 93890 95990 59216 -01
	.23552 22731 54871 76423 -02	.10217 75619 63664 92989 -01
	.16377 94153 15153 96938 -03	.78040 02200 52063 61676 -03
	.43980 12290 60812 99224 -05	.38719 75252 89419 77095 -04
	.11014 77035 81094 26833 -06	.11633 17088 02093 24466 -05
	-.12839 85657 45606 74048 -09	.16457 10741 45580 73556 -07

TABLE B.I (Concluded)

N = 08

NUMERATOR	DENOMINATOR
.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
.19513 36973 48381 21416 +00	.44513 36973 48381 21416 +00
.36028 72455 48033 66895 -01	.91756 59333 63431 14880 -01
.28530 41423 63713 82221 -02	.11479 20657 17017 38378 -01
.21463 38297 93936 23562 -03	.95698 96347 45946 53974 -03
.77757 97235 47219 15163 -05	.54679 34064 85047 10025 -04
.25482 33856 33725 45105 -06	.21015 17140 91386 35547 -05
.27089 12491 61763 77927 -08	.49960 40804 95845 41259 -07
.97825 79525 95621 99712 -11	.56629 36991 72465 57055 -09

N = 09

NUMERATOR	DENOMINATOR
.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
.19823 84189 19627 19265 +00	.44823 84189 19627 19265 +00
.37482 24827 32679 61682 -01	.93986 29744 76192 04289 -01
.31215 54261 10534 67154 -02	.12132 66090 52530 81529 -01
.25168 68581 42141 13445 -03	.10658 74645 55601 67575 -02
.10438 66937 29114 63287 -04	.66311 99549 82139 94997 -04
.41779 89027 92798 76366 -06	.29328 17107 87334 24422 -05
.72282 10040 34696 94233 -08	.89307 30009 59419 32977 -07
.11156 38762 07900 30289 -09	.17075 36697 26887 18576 -08
-.78469 72732 66716 49614 -13	.15708 67540 09828 56485 -10

N = 10

NUMERATOR	DENOMINATOR
.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
.20502 08456 77917 06628 +00	.45502 08456 77917 06628 +00
.39390 75193 16296 55245 -01	.97590 40779 55533 66258 -01
.34858 23655 29237 91179 -02	.13021 15639 98519 94778 -01
.29317 75506 14266 48946 -03	.11999 11137 74704 76100 -02
.13754 73570 29922 39386 -04	.80015 09559 21661 45984 -04
.60964 46174 77455 80030 -06	.39222 83073 88575 92254 -05
.14447 18655 00891 74805 -07	.14003 62118 96032 45150 -06
.30430 04327 31332 24684 -09	.34984 41348 05290 45646 -08
.22059 38908 74765 26250 -11	.55465 89453 73869 45817 -10
.49848 28058 16872 88340 -14	.42591 33901 24021 43020 -12

TABLE B.II

ERROR OF PADÉ APPROXIMATIONS TO E(z)

Let $E_n(z)$ be the n^{th} order main diagonal Padé approximation to E(z), see B.I, and define $\epsilon_n(z) = |E(z) - E_n(z)|$. The tables give $\epsilon_n(z)$ for $n = 2(1)10$, $r = 1(1)10$ and $\theta = 0$, $\pi/2$, π where $z = re^{i\theta}$. The expression in parentheses to the right of each number indicates the power of 10 by which the number is multiplied.

r \ θ	0	π/2	π	π (Relative Error)
n = 2				
1	0.782 (- 5)	0.107 (- 4)	0.320 (- 4)	0.243 (- 4)
2	0.137 (- 3)	0.418 (- 3)	0.232 (- 2)	0.126 (- 2)
3	0.599 (- 3)	0.267 (- 2)	0.433 (- 1)	0.157 (- 1)
4	0.152 (- 2)	0.849 (- 2)	0.455	0.103
5	0.292 (- 2)	0.202 (- 1)	0.288 (1)	0.379
6	0.473 (- 2)	0.361 (- 1)	0.102 (1)	0.732
7	0.686 (- 2)	0.542 (- 1)	0.247 (2)	0.915
8	0.922 (- 2)	0.713 (- 1)	0.533 (2)	0.974
9	0.117 (- 1)	0.845 (- 1)	0.114 (3)	0.991
10	0.143 (- 1)	0.825	0.248 (3)	0.996
n = 3				
1	0.217 (- 7)	0.443 (- 7)	0.101 (- 6)	0.776 (- 7)
2	0.139 (- 5)	0.523 (- 5)	0.302 (- 4)	0.164 (- 4)
3	0.125 (- 4)	0.780 (- 4)	0.128 (- 2)	0.465 (- 3)
4	0.510 (- 4)	0.482 (- 3)	0.252 (- 1)	0.571 (- 2)
5	0.138 (- 3)	0.179 (- 2)	0.345	0.454 (- 1)
6	0.291 (- 3)	0.474 (- 2)	0.454 (1)	0.326
7	0.519 (- 3)	0.985 (- 2)	0.233 (3)	0.863
8	0.823 (- 3)	0.170 (- 1)	0.703 (2)	0.128 (1)
9	0.120 (- 2)	0.255 (- 1)	0.123 (3)	0.107 (1)
10	0.164 (- 2)	0.340 (- 1)	0.254 (3)	0.102 (1)

TABLE B.II (Continued)

r \ θ	0	π/2	π	π (Relative Error)
n = 4				
1	0.593 (-10)	0.128 (- 9)	0.302 (- 9)	0.229 (- 9)
2	0.144 (- 7)	0.610 (- 7)	0.375 (- 6)	0.204 (- 6)
3	0.273 (- 6)	0.209 (- 5)	0.366 (- 4)	0.133 (- 4)
4	0.186 (- 5)	0.236 (- 4)	0.130 (- 2)	0.294 (- 3)
5	0.738 (- 5)	0.141 (- 3)	0.271 (- 1)	0.366 (- 2)
6	0.209 (- 4)	0.558 (- 3)	0.401	0.288 (- 1)
7	0.474 (- 4)	0.162 (- 2)	0.428 (1)	0.159
8	0.920 (- 4)	0.375 (- 2)	0.273 (2)	0.499
9	0.159 (- 3)	0.720 (- 2)	0.945 (2)	0.822
10	0.251 (- 3)	0.119 (- 1)	0.237 (3)	0.952
n = 5				
1	0.837 (-13)	0.187 (-12)	0.451 (-12)	0.342 (-12)
2	0.781 (-10)	0.362 (- 9)	0.227 (- 8)	0.123 (- 8)
3	0.319 (- 8)	0.284 (- 7)	0.502 (- 6)	0.182 (- 6)
4	0.368 (- 7)	0.588 (- 6)	0.316 (- 4)	0.715 (- 5)
5	0.206 (- 6)	0.574 (- 5)	0.102 (- 2)	0.134 (- 3)
6	0.834 (- 6)	0.342 (- 4)	0.217 (- 1)	0.156 (- 2)
7	0.243 (- 5)	0.143 (- 3)	0.355	0.131 (- 1)
8	0.579 (- 5)	0.457 (- 3)	0.513 (1)	0.937 (- 1)
9	0.119 (- 4)	0.117 (- 2)	0.104 (3)	0.904
10	0.218 (- 4)	0.250 (- 2)	0.442 (3)	0.178 (1)
n = 6				
1	0.117 (-15)	0.268 (-15)	0.658 (-15)	0.499 (-15)
2	0.423 (-12)	0.209 (-11)	0.135 (-10)	0.733 (-11)
3	0.376 (-10)	0.373 (- 9)	0.677 (- 8)	0.246 (- 6)
4	0.746 (- 9)	0.139 (- 7)	0.763 (- 6)	0.173 (- 6)
5	0.658 (- 8)	0.217 (- 6)	0.384 (- 4)	0.505 (- 5)
6	0.352 (- 7)	0.192 (- 5)	0.118 (- 2)	0.847 (- 4)
7	0.134 (- 6)	0.113 (- 4)	0.258 (- 1)	0.956 (- 3)
8	0.399 (- 6)	0.489 (- 4)	0.445	0.813 (- 2)
9	0.993 (- 6)	0.165 (- 3)	0.620 (1)	0.539 (- 1)
10	0.215 (- 5)	0.455 (- 3)	0.627 (2)	0.252

TABLE B.II (Continued)

r \ θ	0	π/2	π	π (Relative Error)
n = 7				
1	0.900 (-19)	0.233 (-18)	0.580 (-18)	0.440 (-18)
2	0.140 (-14)	0.722 (-14)	0.472 (-13)	0.256 (-13)
3	0.272 (-12)	0.294 (-11)	0.536 (-10)	0.195 (-10)
4	0.932 (-11)	0.198 (- 9)	0.107 (- 7)	0.242 (- 8)
5	0.125 (- 9)	0.493 (- 8)	0.840 (- 6)	0.110 (- 6)
6	0.925 (- 9)	0.644 (- 7)	0.367 (- 4)	0.263 (- 4)
7	0.462 (- 8)	0.534 (- 6)	0.108 (- 2)	0.400 (- 3)
8	0.173 (- 7)	0.314 (- 5)	0.241 (- 1)	0.440 (- 3)
9	0.524 (- 7)	0.141 (- 4)	0.436	0.379 (- 2)
10	0.134 (- 6)	0.336	0.686 (1)	0.276 (- 1)
n = 8				
1	0.370 (-20)	0.100 (-19)	0.100 (-19)	0.759 (-20)
2	0.459 (-17)	0.246 (-16)	0.164 (-15)	0.890 (-16)
3	0.197 (-14)	0.227 (-13)	0.421 (-12)	0.153 (-12)
4	0.118 (-12)	0.275 (-11)	0.150 (- 9)	0.340 (-10)
5	0.239 (-11)	0.108 (- 9)	0.184 (- 7)	0.242 (- 8)
6	0.249 (-10)	0.207 (- 8)	0.115 (- 5)	0.825 (- 7)
7	0.165 (- 9)	0.238 (- 7)	0.461 (- 4)	0.171 (- 5)
8	0.784 (- 9)	0.187 (- 6)	0.133 (- 2)	0.243 (- 4)
9	0.291 (- 8)	0.109 (- 5)	0.301 (- 1)	0.262 (- 3)
10	0.895 (- 8)	0.500 (- 5)	0.565	0.227 (- 2)
n = 9				
1	0.367 (-20)	0.100 (-19)	0.100 (-19)	0.758 (-20)
2	0.500 (-20)	0.500 (-19)	0.377 (-18)	0.205 (-18)
3	0.950 (-17)	0.117 (-15)	0.217 (-14)	0.788 (-15)
4	0.998 (-15)	0.254 (-13)	0.138 (-11)	0.312 (-12)
5	0.311 (-13)	0.106 (-10)	0.263 (- 9)	0.346 (-10)
6	0.456 (-12)	0.442 (-10)	0.236 (- 7)	0.169 (- 8)
7	0.400 (-11)	0.707 (- 9)	0.127 (- 5)	0.470 (- 7)
8	0.242 (-10)	0.743 (- 8)	0.474 (- 4)	0.866 (- 6)
9	0.111 (- 9)	0.564 (- 7)	0.134 (- 2)	0.117 (- 4)
10	0.408 (- 9)	0.329 (- 6)	0.306 (- 1)	0.123 (- 3)

TABLE B.II (Concluded)

r \ θ	0	π/2	π	π (Relative Error)
1	0.367 (-20)	0.100 (-19)	0.100 (-19)	0.759 (-20)
2	0.551 (-20)	0.141 (-19)	0.200 (-20)	0.109 (-20)
3	0.145 (-18)	0.578 (-18)	0.111 (-16)	0.403 (-17)
4	0.850 (-17)	0.231 (-15)	0.126 (-13)	0.285 (-14)
5	0.407 (-15)	0.227 (-13)	0.375 (-11)	0.493 (-12)
6	0.845 (-14)	0.923 (-12)	0.485 (- 9)	0.341 (-10)
7	0.989 (-13)	0.203 (-10)	0.354 (- 7)	0.131 (- 8)
8	0.766 (-12)	0.284 (- 9)	0.172 (- 5)	0.314 (- 7)
9	0.434 (-11)	0.277 (-8)	0.610 (- 4)	0.530 (- 6)
10	0.193 (-10)	0.204 (- 7)	0.170 (- 2)	0.683 (- 5)

TABLE B.III

COEFFICIENTS OF PADÉ APPROXIMATIONS TO $z^{-1}Si(z)$

Note: For the definition of Si(z) and its Padé approximants, see Section V. The coefficients are given for n = 2(2)10. The expression attached to the numbers on the right indicates the power of 10 by which the numbers are multiplied.

$$\text{e.g., } z^{-1}Si_2(z) = \frac{1 - 0.02555\ldots z^2}{1 + 0.03z^2}$$

	NUMERATOR	DENOMINATOR
N = 02		
	.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
	-.25555 55555 55555 55556 -01	.30000 00000 00000 00000 -01
N = 04		
	.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
	-.30427 89735 63441 27723 -01	.25127 65819 92114 27833 -01
	.51431 13199 43054 89416 -03	.24362 56643 43689 77376 -03
N = 06		
	.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
	-.35595 17688 83526 52167 -01	.19960 37866 72029 03389 -01
	.74030 23630 09289 15834 -03	.18254 56222 98339 34660 -03
	-.44148 31037 35430 42004 -05	.80396 58468 07101 69744 -06
N = 08		
	.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
	-.39097 46707 47744 00193 -01	.16458 08848 07811 55363 -01
	.88244 03534 29201 16961 -03	.13011 19356 94820 91197 -03
	-.75392 62090 91062 85582 -05	.60370 25148 69333 23738 -06
	.24448 76200 37211 27450 -07	.14412 35225 57797 97185 -08

TABLE B.III (Concluded)

N = 10

.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
-.41604 56609 13051 18594 -01	.13950 98946 42504 36962 -01
.98695 05591 58116 83815 -03	.95338 86272 75855 58234 -04
-.99826 03983 42126 42340 -05	.40702 15961 74597 46660 -06
.48154 24911 89950 63472 -07	.11122 28953 69313 70588 -08
-.89943 23724 44804 92329 -10	.16036 02483 25839 25315 -11

TABLE B.IV

ERROR OF PADÉ APPROXIMATION TO $z^{-1}Si(z)$

Let $z^{-1}Si_n(z)$ be the n^{th} order main diagonal (see B.III) Padé approximant to $z^{-1}Si(z)$ and define $\epsilon_n(z) = |z^{-1}Si(z)-z^{-1}Si_n(z)|$. The tables give $\epsilon_n(z)$ for $n = 2(2)10$ and $z = 1(1)10$, z real. The expression in parentheses to the right of each number indicates the power of 10 by which the number is multiplied.

z \ n	2	4	6	8	10
1	0.210 (-4)	0.142 (-8)	0.251 (-13)	0.170 (-18)	0.100 (-20)
2	0.112 (-2)	0.127 (-5)	0.369 (- 9)	0.392 (-13)	0.186 (-17)
3	0.990 (-2)	0.590 (-4)	0.898 (- 7)	0.498 (-10)	0.122 (-13)
4	0.401 (-1)	0.775 (-3)	0.392 (- 5)	0.714 (- 8)	0.570 (-11)
5	0.104	0.496 (-2)	0.647 (- 4)	0.302 (- 6)	0.616 (- 9)
6	0.234	0.198 (-1)	0.565 (- 3)	0.579 (- 5)	0.254 (- 7)
7	0.310	0.563 (-1)	0.313 (- 2)	0.630 (- 4)	0.539 (- 6)
8	0.415	0.125	0.123 (- 1)	0.449 (- 3)	0.692 (- 5)
9	0.497	0.227	0.370 (- 1)	0.229 (- 2)	0.599 (- 4)
10	0.555	0.335	0.896 (- 1)	0.889 (- 2)	0.376 (- 3)

TABLE B.V

COEFFICIENTS OF PADÉ APPROXIMATIONS TO $4z^{-2}H(z)$

Note: For the definition of $H(z)$ and its Padé approximants, see Section V. The coefficients are given for $n = 2(2)10$. The expression attached to the numbers on the right indicates the power of 10 by which the numbers are multiplied.

$$\text{e.g., } 4z^{-2}H_2(z) = \frac{1 - 0.019444\ldots z^2}{1 + 0.0222\ldots z^2}$$

N = 02

NUMERATOR	DENOMINATOR
.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
-.19444 44444 44444 44444 -01	.22222 22222 22222 22222 -01

N = 04

NUMERATOR	DENOMINATOR
.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
-.21441 94756 55430 71161 -01	.20224 71910 11235 95506 -01
.23504 84513 40586 17205 -03	.15181 91546 28143 39219 -03

N = 06

NUMERATOR	DENOMINATOR
.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
-.24662 60143 38362 60391 -01	.17004 06523 28304 06275 -01
.34682 10542 75433 98340 -03	.12939 78463 84108 31895 -03
-.15875 89095 57136 15182 -05	.46027 66426 80130 08786 -06

N = 08

NUMERATOR	DENOMINATOR
.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
-.27191 18247 38415 96633 -01	.14475 48419 28250 70034 -01
.42197 89824 50745 02333 -03	.99198 23122 58636 82145 -04
-.27389 37015 82204 22581 -05	.39189 01649 14937 85289 -06
.70221 55017 22417 74512 -08	.77891 33652 63414 35304 -09

TABLE B.V (Concluded)

N = 10

NUMERATOR	DENOMINATOR
.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
-.29128 03421 36361 44413 -01	.12538 63245 30305 22254 -01
.47973 22804 11881 21947 -03	.76249 37336 22270 54117 -04
-.36817 31277 22802 06720 -05	.28627 47330 74607 14153 -06
.14091 69854 52941 91735 -07	.67825 76089 43662 46538 -09
-.21678 54057 76489 52502 -10	.83325 19383 71648 11691 -12

TABLE B.VI

ERROR OF PADÉ APPROXIMATION TO $4z^{-2}H(z)$

Let $4z^{-2}H_n(z)$ be the n^{th} order main diagonal (see B.V) Padé approximant to $4z^{-2}H(z)$ and define $\epsilon_n(z) = |4z^{-2}||H(z)-H_n(z)|$. The tables give $\epsilon_n(z)$ for $n = 2(2)10$ and $z = 1(1)10$. The expression in parentheses to the right of each number indicates the power of 10 by which the number is multiplied.

z \ n	2	4	6	8	10
1	0.584 (-5)	0.336 (-9)	0.433 (-14)	1.000 (-20)	1.000 (-20)
2	0.443 (-3)	0.308 (-6)	0.646 (-10)	0.535 (-14)	0.210 (-18)
3	0.414 (-2)	0.148 (-4)	0.161 (- 7)	0.691 (-11)	0.138 (-14)
4	0.179 (-1)	0.205 (-3)	0.728 (- 6)	0.102 (- 8)	0.657 (-12)
5	0.599 (-1)	0.140 (-2)	0.126 (- 4)	0.443 (- 7)	0.721 (-10)
6	0.104	0.597 (-2)	0.116 (- 3)	0.881 (- 6)	0.308 (- 8)
7	0.177	0.184 (-1)	0.680 (- 3)	0.100 (- 4)	0.675 (- 7)
8	0.341	0.441 (-1)	0.286 (- 2)	0.751 (- 4)	0.900 (- 6)
9	0.339	0.873 (-1)	0.920 (- 2)	0.404 (- 3)	0.812 (- 5)
10	0.410	0.149	0.240 (- 1)	0.167 (- 2)	0.535 (- 4)

TABLE B.VII

COEFFICIENTS OF RATIONAL APPROXIMATIONS TO $E(z)$

Here we present coefficients of the numerator and denominator polynomials of the rational approximation to $E(z)$ defined in (5.11). We give coefficients for $n = 0(1)10$. The sequence of numbers given is for the lowest power to the highest power, respectively. The expression to the right of each number is the power of 10 by which it is multiplied.

$$\text{e.g., } E_2(z) = \frac{3z + 36}{z^2+12z+36}$$

N	NUMERATOR		DENOMINATOR	
N = 00				
	.10000 00000 00000 00000	01	.10000 00000 00000 00000	01
N = 01				
	.40000 00000 00000 00000	01	.40000 00000 00000 00000	01
	.00000 00000 00000 00000	00	.10000 00000 00000 00000	01
N = 02				
	.36000 00000 00000 00000	02	.36000 00000 00000 00000	02
	.30000 00000 00000 00000	01	.12000 00000 00000 00000	02
	.00000 00000 00000 00000	00	.10000 00000 00000 00000	01
N = 03				
	.48000 00000 00000 00000	03	.48000 00000 00000 00000	03
	.60000 00000 00000 00000	02	.18000 00000 00000 00000	03
	.56666 66666 66666 66667	01	.24000 00000 00000 00000	02
	.00000 00000 00000 00000	00	.10000 00000 00000 00000	01

TABLE B.VII (Continued)

	NUMERATOR	DENOMINATOR
N = 04		
	.84000 00000 00000 00000 +04	.84000 00000 00000 00000 +04
	.12600 00000 00000 00000 +04	.33600 00000 00000 00000 +04
	.16666 66666 66666 66667 +03	.54000 00000 00000 00000 +03
	.41666 66666 66666 66667 +01	.40000 00000 00000 00000 +02
	.00000 00000 00000 00000 -99	.10000 00000 00000 00000 +01
N = 05		
	.18144 00000 00000 00000 +06	.18144 00000 00000 00000 +06
	.30240 00000 00000 00000 +05	.75600 00000 00000 00000 +05
	.46200 00000 00000 00000 +04	.13440 00000 00000 00000 +05
	.21000 00000 00000 00000 +03	.12600 00000 00000 00000 +04
	.65666 66666 66666 66667 +01	.60000 00000 00000 00000 +02
	.00000 00000 00000 00000 -99	.10000 00000 00000 00000 +01
N = 06		
	.46569 60000 00000 00000 +07	.46569 60000 00000 00000 +07
	.83160 00000 00000 00000 +06	.19958 40000 00000 00000 +07
	.13776 00000 00000 00000 +06	.37800 00000 00000 00000 +06
	.81900 00000 00000 00000 +04	.40320 00000 00000 00000 +05
	.41160 00000 00000 00000 +03	.25200 00000 00000 00000 +04
	.49000 00000 00000 00000 +01	.84000 00000 00000 00000 +02
	.00000 00000 00000 00000 -99	.10000 00000 00000 00000 +01
N = 07		
	.13837 82400 00000 00000 +09	.13837 82400 00000 00000 +09
	.25945 92000 00000 00000 +08	.60540 48000 00000 00000 +08
	.45276 00000 00000 00000 +07	.11975 04000 00000 00000 +08
	.31416 00000 00000 00000 +06	.13860 00000 00000 00000 +07
	.19580 40000 00000 00000 +05	.10080 00000 00000 00000 +06
	.46480 00000 00000 00000 +03	.45360 00000 00000 00000 +04
	.71857 14285 71428 57143 +01	.11200 00000 00000 00000 +03
	.00000 00000 00000 00000 -99	.10000 00000 00000 00000 +01

TABLE B.VII (Concluded)

N = 08

NUMERATOR	DENOMINATOR
.46702 65600 00000 00000 +10	.46702 65600 00000 00000 +10
.90810 72000 00000 00000 +09	.20756 73600 00000 00000 +10
.16432 41600 00000 00000 +09	.42378 33600 00000 00000 +09
.12612 60000 00000 00000 +08	.51891 84000 00000 00000 +08
.89073 60000 00000 00000 +06	.41580 00000 00000 00000 +07
.29106 00000 00000 00000 +05	.22176 00000 00000 00000 +06
.78274 28571 42857 14286 +03	.75600 00000 00000 00000 +04
.54357 14285 71428 57143 +01	.14400 00000 00000 00000 +03
.00000 00000 00000 00000 -99	.10000 00000 00000 00000 +01

N = 09

NUMERATOR	DENOMINATOR
.17643 22560 00000 00000 +12	.17643 22560 00000 00000 +12
.35286 45120 00000 00000 +11	.79394 51520 00000 00000 +11
.65585 52000 00000 00000 +10	.16605 38880 00000 00000 +11
.54054 00000 00000 00000 +09	.21189 16800 00000 00000 +10
.41441 40000 00000 00000 +08	.18162 14400 00000 00000 +09
.16336 32000 00000 00000 +07	.10810 80000 00000 00000 +08
.57052 28571 42857 14286 +05	.44352 00000 00000 00000 +06
.83442 85714 28571 42857 +03	.11880 00000 00000 00000 +05
.76579 36507 93650 79365 +01	.18000 00000 00000 00000 +03
.00000 00000 00000 00000 -99	.10000 00000 00000 00000 +01

N = 10

NUMERATOR	DENOMINATOR
.73748 68300 80000 00000 +13	.73748 68300 80000 00000 +13
.15084 95788 80000 00000 +13	.33522 12864 00000 00000 +13
.28621 23264 00000 00000 +12	.71455 06368 00000 00000 +12
.24872 04720 00000 00000 +11	.94097 20320 00000 00000 +11
.20211 87168 00000 00000 +10	.84756 67200 00000 00000 +10
.90210 12000 00000 00000 +08	.54486 43200 00000 00000 +09
.36810 65142 85714 28571 +07	.25225 20000 00000 00000 +08
.76668 42857 14285 71429 +05	.82368 00000 00000 00000 +06
.12887 46031 74603 17460 +04	.17820 00000 00000 00000 +05
.58579 36507 93650 79365 +01	.22000 00000 00000 00000 +03
.00000 00000 00000 00000 -99	.10000 00000 00000 00000 +01

TABLE B.VIII

ERROR OF RATIONAL APPROXIMATION TO $E(z)$

Let $E_n(z)$ be the n^{th} order rational approximation to $E(z)$ as defined in (5.11) and Table B.VII. Set $\epsilon_n(z) = | E(z)-E_n(z) |$. The tables give $\epsilon_n(z)$ for $n = 3(1)10$, $r = 1(1)10$ and $\theta = 0, \pi/2, \pi$ where $z = re^{i\theta}$. The expression in parentheses to the right of each number indicates the power of 10 by which the number is multiplied.

$r \backslash \theta$	0	$\pi/2$	π	π (Relative Error)
n = 3				
1	0.593 (-5)	0.177 (-4)	0.487 (-4)	0.369 (-4)
2	0.272 (-4)	0.291 (-3)	0.219 (-2)	0.119 (-2)
3	0.134 (-4)	0.148 (-2)	0.317 (-1)	0.115 (-1)
4	0.870 (-4)	0.444 (-2)	0.284	0.643 (-1)
5	0.286 (-3)	0.975 (-2)	0.175 (1)	0.230
6	0.566 (-3)	0.172 (-1)	0.719 (1)	0.517
7	0.895 (-3)	0.259 (-1)	0.206 (2)	0.762
8	0.125 (-2)	0.345 (-1)	0.490 (2)	0.896
9	0.160 (-2)	0.413 (-1)	0.110 (3)	0.957
10	0.193 (-2)	0.454 (-1)	0.243 (3)	0.976
n = 4				
1	0.188 (-6)	0.287 (-6)	0.754 (-6)	0.571 (-6)
2	0.407 (-5)	0.543 (-5)	0.703 (-4)	0.382 (-4)
3	0.226 (-4)	0.155 (-4)	0.180 (-2)	0.655 (-3)
4	0.716 (-4)	0.198 (-3)	0.276 (-1)	0.624 (-2)
5	0.166 (-3)	0.102 (-2)	0.326	0.429 (-1)
6	0.315 (-3)	0.317 (-2)	0.374 (1)	0.269
7	0.524 (-3)	0.724 (-2)	0.274 (3)	0.101 (2)
8	0.787 (-3)	0.133 (-1)	0.773 (2)	0.141 (1)
9	0.110 (-2)	0.209 (-1)	0.126 (3)	0.110 (1)
10	0.145 (-2)	0.288 (-1)	0.256 (3)	0.103 (1)

TABLE B.VIII (Continued)

$r \backslash \theta$	0	$\pi/2$	π	π (Relative Error)
n = 5	0			
1	0.376 (-8)	0.632 (-8)	0.125 (-7)	0.947 (-8)
2	0.140 (-6)	0.379 (-6)	0.188 (-5)	0.102 (-5)
3	0.918 (-6)	0.449 (-5)	0.674 (-4)	0.245 (-4)
4	0.290 (-5)	0.306 (-4)	0.148 (-2)	0.335 (-3)
5	0.598 (-5)	0.145 (-3)	0.242 (-1)	0.318 (-2)
6	0.916 (-5)	0.504 (-3)	0.314	0.226 (-1)
7	0.107 (-4)	0.136 (-2)	0.318 (1)	0.118
8	0.860 (-5)	0.298 (-2)	0.216 (2)	0.395
9	0.117 (-5)	0.553 (-2)	0.848 (2)	0.737
10	0.129 (-4)	0.895 (-2)	0.228 (3)	0.916
n = 6				
1	0.760 (-10)	0.125 (-9)	0.225 (-9)	0.170 (-9)
2	0.602 (-8)	0.154 (-7)	0.559 (-7)	0.304 (-7)
3	0.665 (-7)	0.253 (-6)	0.233 (-5)	0.847 (-6)
4	0.337 (-6)	0.184 (-5)	0.603 (-4)	0.136 (-4)
5	0.113 (-5)	0.883 (-5)	0.129 (-2)	0.170 (-3)
6	0.295 (-5)	0.290 (-4)	0.229 (-1)	0.165 (-2)
7	0.649 (-5)	0.936 (-4)	0.343	0.127 (-1)
8	0.126 (-4)	0.288 (-3)	0.465 (1)	0.850
9	0.221 (-4)	0.778 (-3)	0.825 (2)	0.717
10	0.360 (-4)	0.178 (-2)	0.508 (3)	0.204
n = 7				
1	0.144 (-11)	0.236 (-11)	0.407 (-11)	0.308 (-11)
2	0.229 (- 9)	0.579 (- 9)	0.185 (- 8)	0.101 (- 8)
3	0.370 (- 8)	0.137 (- 7)	0.936 (- 7)	0.340 (- 7)
4	0.236 (- 7)	0.113 (- 6)	0.240 (- 5)	0.543 (- 6)
5	0.905 (- 7)	0.526 (- 6)	0.551 (- 4)	0.725 (- 5)
6	0.251 (- 6)	0.142 (- 5)	0.120 (- 2)	0.863 (- 4)
7	0.558 (- 6)	0.627 (- 5)	0.229 (- 1)	0.848 (- 3)
8	0.105 (- 5)	0.341 (- 4)	0.367	0.671 (- 2)
9	0.172 (- 5)	0.130 (- 3)	0.493 (1)	0.429 (- 1)
10	0.255 (- 5)	0.380 (- 3)	0.506 (2)	0.203

TABLE B.VIII (Concluded)

r \ e	0	π/2	π	π (Relative Error)
n = 8				
1	0.259 (-13)	0.420 (-13)	0.712 (-13)	0.539 (-13)
2	0.825 (-11)	0.208 (-10)	0.626 (-10)	0.340 (-10)
3	0.201 (- 9)	0.759 (- 9)	0.431 (- 8)	0.157 (- 8)
4	0.173 (- 8)	0.939 (- 8)	0.116 (- 6)	0.262 (- 7)
5	0.853 (- 8)	0.648 (- 7)	0.237 (- 5)	0.312 (- 6)
6	0.297 (- 7)	0.327 (- 6)	0.528 (- 4)	0.380 (- 5)
7	0.820 (- 7)	0.142 (- 5)	0.120 (- 2)	0.444 (- 4)
8	0.192 (- 6)	0.555 (- 5)	0.242 (- 1)	0.442 (- 2)
9	0.399 (- 6)	0.189 (- 4)	0.418	0.363 (- 2)
10	0.754 (- 6)	0.558 (- 4)	0.638 (1)	0.256 (- 1)
n = 9				
1	0.439 (-15)	0.709 (-15)	0.119 (-14)	0.902 (-15)
2	0.280 (-12)	0.708 (-12)	0.206 (-11)	0.112 (-11)
3	0.102 (-10)	0.391 (-10)	0.205 (- 9)	0.745 (-10)
4	0.117 (- 9)	0.655 (- 9)	0.657 (- 8)	0.149 (- 8)
5	0.710 (- 9)	0.571 (- 8)	0.127 (- 6)	0.167 (- 7)
6	0.290 (- 8)	0.330 (- 7)	0.235 (- 5)	0.169 (- 6)
7	0.904 (- 8)	0.142 (- 6)	0.535 (- 4)	0.198 (- 5)
8	0.231 (- 7)	0.474 (- 6)	0.126 (- 2)	0.230 (- 4)
9	0.509 (- 7)	0.124 (- 5)	0.266 (- 1)	0.231 (- 3)
10	0.995 (- 7)	0.283 (- 5)	0.484	0.194 (- 2)
n = 10				
1	0.658 (-17)	0.114 (-16)	0.188 (-16)	0.142 (-16)
2	0.898 (-14)	0.227 (-13)	0.646 (-13)	0.351 (-13)
3	0.492 (-12)	0.189 (-11)	0.952 (-11)	0.346 (-11)
4	0.747 (-11)	0.425 (-10)	0.392 (- 9)	0.887 (-10)
5	0.566 (-10)	0.463 (- 9)	0.831 (- 8)	0.109 (- 8)
6	0.267 (- 9)	0.315 (- 8)	0.133 (- 6)	0.956 (- 8)
7	0.101 (- 8)	0.151 (- 7)	0.241 (- 5)	0.893 (- 7)
8	0.296 (- 8)	0.546 (- 7)	0.569 (- 4)	0.104 (- 5)
9	0.742 (- 8)	0.154 (- 6)	0.139 (- 2)	0.121 (- 4)
10	0.164 (- 7)	0.426 (- 6)	0.256 (- 1)	0.103 (- 3)

APPENDIX C

FORTRAN PROGRAM FOR COMPUTING RATIONAL APPROXIMATIONS

Here we describe a FORTRAN program which computes the rational approximations (2.1) and (2.18) for the incomplete gamma function and its special cases. The selection of input data determines the function to be approximated. We also include a description of input data, operating procedures, output, and a listing of the FORTRAN program.

The input data are read in the order r, θ, A, B, NCODE, IOPT, LOPT, and $\left\{\begin{matrix} n \\ \text{or} \\ \text{Error} \end{matrix}\right\}$.

The value IOPT determines which value in the braces $\left\{\begin{matrix} n \\ \text{or} \\ \text{Error} \end{matrix}\right\}$ should be read. If the n^{th} approximant is desired, IOPT = 7 is entered and the corresponding value in braces, n, is entered. IOPT $\neq$ 7 instructs the computer to iterate until an integer m is found such that $|\nu^{-1} z^{\nu} e^{z}| |V_m(z,\nu) - V_{m-1}(z,\nu)| <$ Error , and therefore Error should accompany this choice of IOPT. Selection of LOPT offers the choice of computing either $V_n(z,\nu)$ or $S_n(z,\nu)$ (see (2.1) and (2.18)). If LOPT = 9 , the rational approximate is $V_n(z,\nu)$ and if LOPT $\neq$ 9 , the approximate is $S_n(z,\nu)$.

For $z = re^{i\theta}$ (θ in degrees) and $\nu = A+iB$, the following table indicates the values of ν and NCODE to select to compute the approximations to the designated functions. A listing of the FORTRAN program concludes this Appendix.

NCODE	A	B	Function	Expressions Computed	Output
1	A	B	$\int_0^z t^{\nu-1}e^{t}\,dt = \nu^{-1}z^{\nu}e^{z}\left\{V_n(z,\nu) + R_n(z,\nu)\right\}$, $\mathrm{Re}(\nu) > 0$	$V_n^*(z,\nu) = \nu^{-1}z^{\nu}e^{+z}V_n(z,\nu)$	$\mathrm{Re}\left[V_n^*(z,\nu)\right], \mathrm{Im}\left[V_n^*(z,\nu)\right]$
2	0.5	0	$\int_0^z e^{-t^2}dt = ze^{-z^2}\left\{V_n(z^2e^{-i\pi},\tfrac{1}{2}) + R_n(z^2e^{-i\pi},\tfrac{1}{2})\right\}$	$\overset{*}{\mathrm{Erf}}(z) = ze^{-z^2}V_n(z^2e^{-i\pi},\tfrac{1}{2})$ and $\overset{*}{\mathrm{Erfc}}(z) = \tfrac{1}{2}\pi^{\frac{1}{2}} - \overset{*}{\mathrm{Erf}}(z)$	$\mathrm{Re}\left[\overset{*}{\mathrm{Erf}}(z)\right], \mathrm{Im}\left[\overset{*}{\mathrm{Erf}}(z)\right]$ $\mathrm{Re}\left[\overset{*}{\mathrm{Erfc}}(z)\right], \mathrm{Im}\left[\overset{*}{\mathrm{Erfc}}(z)\right]$
3	0.5	0	$\int_0^z e^{-t^2/2}dt = ze^{-z^2/2}\left\{V_n(z^2e^{-i\pi/2},\tfrac{1}{2}) + R_n(z^2e^{-i\pi/2},\tfrac{1}{2})\right\}$	$a(z) = ze^{-z^2/2}V_n(z^2e^{-i\pi/2},\tfrac{1}{2})$ and $a^*(z) = (\pi/2)^{\frac{1}{2}} - a(z)$	$\mathrm{Re}\left[a(z)\right], \mathrm{Im}\left[a(z)\right]$ $\mathrm{Re}\left[a^*(z)\right], \mathrm{Im}\left[a^*(z)\right]$
4	0.5	0	$\mathcal{C}(z) = \int_0^z t^{-\frac{1}{2}}\cos t\,dt = \tfrac{1}{2}(J_1+J_2)$, $\mathcal{S}(z) = \int_0^z t^{-\frac{1}{2}}\sin t\,dt = \frac{1}{2i}(J_1-J_2)$, $C(z) = \int_z^\infty t^{-\frac{1}{2}}\cos t\,dt = (\pi/2)^{\frac{1}{2}} - \mathcal{C}(z)$, and $S(z) = \int_z^\infty t^{-\frac{1}{2}}\sin t\,dt = (\pi/2)^{\frac{1}{2}} - \mathcal{S}(z)$	$J_1^* = 2z^{\frac{1}{2}}e^{iz}V_n(\tfrac{1}{2},iz)$, $J_2^* = 2z^{\frac{1}{2}}e^{-iz}V_n(\tfrac{1}{2},-iz)$, $\mathcal{C}^*(z) = \tfrac{1}{2}(J_1^*+J_2^*)$, $\mathcal{S}^*(z) = \frac{1}{2i}(J_1^*-J_2^*)$,	$\mathrm{Re}\left[\mathcal{C}^*(z)\right], \mathrm{Im}\left[\mathcal{C}^*(z)\right]$ $\mathrm{Re}\left[\mathcal{S}^*(z)\right], \mathrm{Im}\left[\mathcal{S}^*(z)\right]$ $\mathrm{Re}\left[C^*(z)\right], \mathrm{Im}\left[C^*(z)\right]$ $\mathrm{Re}\left[S^*(z)\right], \mathrm{Im}\left[S^*(z)\right]$

NCODE	A	B	Function	Expressions Computed	Output
4	0.5	0	where		
(concluded)			$J_1 = 2z^{\frac{1}{2}}e^{iz}\left\{V_n(ze^{i\pi/2},\tfrac{1}{2})+R_n(ze^{i\pi/2},\tfrac{1}{2})\right\}$	$C^*(z) = (\pi/2)^{\frac{1}{2}} - \mathcal{C}^*(z)$	
			and		
			$J_2 = 2z^{\frac{1}{2}}e^{-iz}\left\{V_n(ze^{-i\pi/2},\tfrac{1}{2})+R_n(ze^{-i\pi/2},\tfrac{1}{2})\right\}$	$S^*(z) = (\pi/2)^{\frac{1}{2}} - \mathcal{S}^*(z)$	
5	0.5	0	$\mathcal{C}(z) = \int_0^z \cos t^2 dt = \tfrac{1}{2}(K_1+K_2)$,	$K_1^* = ze^{iz^2}\ V_n(ze^{i\pi/2},\tfrac{1}{2})$,	$\mathrm{Re}\left[\mathcal{C}^*(z)\right]$, $\mathrm{Im}\left[\mathcal{C}^*(z)\right]$
			$\mathcal{S}(z) = \int_0^z \sin t^2 dt = \frac{1}{2i}\ (K_1-K_2)$,	$K_2^* = ze^{-iz^2}V_n(ze^{-i\pi/2},\tfrac{1}{2})$,	$\mathrm{Re}\left[\mathcal{S}^*(z)\right]$, $\mathrm{Im}\left[\mathcal{S}^*(z)\right]$
			$F(z) = \int_z^\infty \cos t^2 dt = \tfrac{1}{2}(\pi/2)^{\frac{1}{2}} - \mathcal{C}(z)$,	$\mathcal{C}^*(z) = \tfrac{1}{2}(K_1^*+K_2^*)$,	$\mathrm{Re}\left[F^*(z)\right]$, $\mathrm{Im}\left[F^*(z)\right]$
			and		
			$L(z) = \int_z^\infty \sin t^2 dt = \tfrac{1}{2}(\pi/2)^{\frac{1}{2}} - \mathcal{S}(z)$	$\mathcal{S}^*(z) = \frac{1}{2i}\ (K_1^*-K_2^*)$,	$\mathrm{Re}\left[L^*(z)\right]$, $\mathrm{Im}\left[L^*(z)\right]$
			where	$F^*(z) = \tfrac{1}{2}(\pi/2)^{\frac{1}{2}} - \mathcal{C}^*(z)$, and	
			$K_1 = ze^{iz^2}\left\{V_n(z^2e^{i\pi/2},\tfrac{1}{2})+R_n(z^2e^{i\pi/2},\tfrac{1}{2})\right\}$	$L^*(z) = \tfrac{1}{2}(\pi/2)^{\frac{1}{2}} - \mathcal{S}^*(z)$	
			and		
			$K_2 = ze^{-iz^2}\left\{V_n(z^2e^{-i\pi/2},\tfrac{1}{2})+R_n(z^2e^{-i\pi/2},\tfrac{1}{2})\right\}$		

```
C       RATIONAL APPROX.,ASCENDING SERIES
        DIMENSION VR(40),VI(40)
 1      READ 400,R,TH,A,B,NCODE,IOPT,KOPT,LOPT,
        PRINT 403,R,TH,A,B
        PI=3.1415926
        THA=TH*PI/180.
        HPI=.5*PI
        RTPI=1.7724538
        NN=1
        X=R*COSF(THA)
        Y=R*SINF(THA)
        RO=LOGF(R)
 203    IF (IOPT-7) 101,102,101
 101    READ 401,ERROR
        ERROR=ERROR*ERROR
        GO TO 103
 102    READ 402,NT
        MT=NT+1
 103    GO TO (70,80,90,100,110),NCODE
 90     CM=.5
        GO TO 81
 80     CM=1.0
 81     RZ=(Y*Y-X*X)*CM
        RZI=-2.*X*Y*CM
        TH=THA+RZI
        C=COSF(TH)
        D=SINF(TH)
        DEN=R*EXPF(RZ)
        CR=C*DEN
        CI=D*DEN
        X=RZ
        Y=RZI
        GO TO 300
 70     C=A*RO-B*THA+X
        DN=1./(A*A+B*B)
        TH=B*RO+A*THA+Y
        DN=EXPF(C)*DN
        CR=DN*(A*COSF(TH)+B*SINF(TH))
        CI=DN*(A*SINF(TH)-B*COSF(TH))
        GO TO 300
 100    THA=.5*THA
        C=X
        X=-Y
        Y=C
 105    DN=SQRTF(R)*EXPF(X)
        CR=DN*COSF(THA+Y)
        CI=DN*SINF(THA+Y)
        GO TO 300
 110    C=X*X-Y*Y
        D=-2.*X*Y
```

```
      Y=C
      X=D
106   DN=R*EXPF(X)
      CR=DN*COSF(THA+Y)
      CI=DN*SINF(THA+Y)
      GO TO 300
60    IF (NN-2) 61,62,62
61    NN=2
      X=-X
      Y=-Y
      CC=C
      DD=D
      IF(NCODE-4)106,105,106
300   IF (LOPT-9) 202,205,202
202   A=A-1.
205   AI=(A+1.)*(A+1.)+B*B
      AI=1./AI
      IF(LOPT-9)11,10,11
10    P1=1.
      S2=A+2.+X
      S1=1.
      P2=(A+2.)-((A+1.)*X+B*Y)*AI
      Q1=0.
      Q2=B+(B*X-Y*(A+1.))*AI
      T1=0.
      T2=B+Y
      GO TO 301
11    P1=0.0
      P2=A+2.
      Q1=0.0
      Q2=B
      S1=1.0
      S2=X+A+2.
      T1=0.0
      T2=Y+B
301   J=1
      VR(1)=(P2*S2+Q2*T2)/(S2*S2+T2*T2)
      VI(1)=(Q2*S2-P2*T2)/(S2*S2+T2*T2)
302   K=J+1
      F=J
      G=K
      H=2.*F+A
      AL=H*(G+A)-B*B
      BL=B*(H+G+A)
      C =(H+1.)*(H*(H+2.)-B*B+A*X-B*Y)
      C=C-B*(B*X+A*Y+B*(2.*H+2.))
      D=B*(H*(H+2.)-B*B+A*X-B*Y)
      D=D+(H+1.)*(B*X+A*Y+B*(2.*H+2.))
      E=F*((X*X-Y*Y)*(H+2.)-2.*B*X*Y)
      FE=F*(2.*X*Y*(H+2.)+B*(X*X-Y*Y))
      DN=1./(AL*AL+BL*BL)
```

```
      ACPBD=AL*C+BL*D
      BCMAD=BL*C-AL*D
      AEPBF=AL*E+BL*FE
      BEMAF=BL*E-AL*FE
      P3=DN*(P2*ACPBD+Q2*BCMAD+P1*AEPBF+Q1*BEMAF)
      Q3=DN*(-P2*BCMAD+Q2*ACPBD-P1*BEMAF+Q1*AEPBF)
      S3=DN*(S2*ACPBD+T2*BCMAD+S1*AEPBF+T1*BEMAF)
      T3=DN*(-S2*BCMAD+T2*ACPBD-S1*BEMAF+T1*AEPBF)
      Q1=Q2
      Q2=Q3
      P1=P2
      P2=P3
      S1=S2
      S2=S3
      T1=T2
      T2=T3
      L=K+1
      IF (IOPT-7) 304,303,304
303   IF (NT-L) 304,304,305
305   J=J+1
      GO TO 302
304   DEN=1./(S2*S2+T2*T2)
      VR(K)=(P2*S2+Q2*T2)*DEN
      VI(K)=(Q2*S2-P2*T2)*DEN
306   IF (IOPT-7) 308,307,308
307   IF (NT-K) 308,308,305
308   RER=VR(K)-VR(K-1)
      REI=VI(K)-VI(K-1)
      IF (IOPT-7) 309,310,309
309   C=CR*RER-CI*REI
      D=CI*RER+CR*REI
      ERT=C*C+D*D
      IF (ERT-ERROR) 310,310,305
310   C=CR*VR(K)-CI*VI(K)
      D=CI*VR(K)+CR*VI(K)
      GO TO (30,40,50,60,60),NCODE
30    PRINT 500,
      PRINT 501,K,C,D
      GO TO 68
40    PRINT 502,
      PRINT 501,K,C,D
      C=.88622692-C
      D=-D
      PRINT 501,K,C,D
      GO TO 68
50    PRINT 502,
      PRINT 501,K,C,D
      C=1.2533141-C
      D=-D
      PRINT 501,K,C,D
      GO TO 68
```

```
62    CM=1.
      IF (NCODE-4) 64,65,64
64    CM=.5
65    SIZR=CM*(DD-D)
      SIZI=CM*(C-CC)
      CIZR=CM*(C+CC)
      CIZI=CM*(D+DD)
      PRINT 505,
      PRINT 506,K,SIZR,SIZI,CIZR,CIZI
      DN=1.2533141
      IF (NCODE-5) 66,67,66
67    DN=.5*DN
66    CIZR=DN-CIZR
      SIZR=DN-SIZR
      CIZI=-CIZI
      SIZI=-SIZI
      PRINT 506,K,SIZR,SIZI,CIZR,CIZI
68    IF (SENSE SWITCH 1) 69,1
69    PRINT 501,K,VR(K),VI(K)
      GO TO 1
400   FORMAT(4F12.0,4I3)
401   FORMAT(E14.7)
402   FORMAT(I3)
403   FORMAT(4(E14.7,2X)/)
500   FORMAT(15H INC. GAMMA FN./)
501   FORMAT(I3,1X,2(E14.7,2X)/)
502   FORMAT(15H ERROR FUNCTION/)
505   FORMAT(18H FRESNEL INTEGRALS/)
506   FORMAT(I3,2X,4(E14.7,2X)/)
      END
```

APPENDIX D

FORTRAN PROGRAM FOR COMPUTATION OF THE EXPONENTIAL AND CIRCULAR FUNCTIONS

The following FORTRAN program is based on the results of Section IV.

Input: (Cards) Let n be the order of the rational approximations to the exponential and circular functions, and suppose that these approximations are desired for $n = n_1(1)n_k$ and $x = x_1(\Delta x)x_m$. The data are entered in the order n_1 , n_k , x_1 , x_m , Δx .

Switch settings: None.

Output: (Printed) The output is of the following form for $x_1, x_2, \ldots, x_m$.

$$e_{n_1}^{(x_1)} \text{ , } \sin_{n_1}(x_1) \text{ , } \cos_{n_1}(x_1) \text{ , } \tan_{n_1}(x_1)$$

$$e_{n_2}^{(x_1)} \text{ , } \sin_{n_2}(x_1) \text{ , } \cos_{n_2}(x_1) \text{ , } \tan_{n_2}(x_1)$$

- -

$$e_{n_k}^{(x_1)} \text{ , } \sin_{n_k}(x_1) \text{ , } \cos_{n_k}(x_1) \text{ , } \tan_{n_k}(x_1)$$

```
C      RATIONAL APPROX. TO EXP(-Z),COS(Z),SIN(Z),TAN(Z)
 1     READ 100,NI,NF,ZI,ZF,ZD
       PRINT 200
       Z=ZI
 2     ZP=Z*Z
       ZM=-ZP
       N=2
       FM=2.0
       CM=2.*(2.*FM+1.)
       A1P=2.0
       A2P=12.0+ZP
       B1P=1.0
       B2P=6.0
       A1M=2.0
       A2M=12.0+ZM
       B1M=1.0
       B2M=6.0
 3     N=N+1
       A3P=CM*A2P+ZP*A1P
       A3M=CM*A2M+ZM*A1M
       B3P=CM*B2P+ZP*B1P
       B3M=CM*B2M+ZM*B1M
       A1P=A2P
       A2P=A3P
       B1P=B2P
       B2P=B3P
       A1M=A2M
       A2M=A3M
       B1M=B2M
       B2M=B3M
       FM=FM+1.0
       CM=2.*(2.*FM+1.0)
       IF (NI-N) 4,4,3
 4     EZ=(A3P-Z*B3P)/(A3P+Z*B3P)
       D=A3M*A3M+ZP*B3M*B3M
       CZ=(A3M*A3M+ZM*B3M*B3M)/D
       SZ=2.*Z*A3M*B3M/D
       D=SZ/CZ
       PRINT 300,N,Z,EZ,SZ,CZ,D
       IF (NF-N) 1,5,3
 5     IF (ZF-Z) 1,1,6
 6     Z=Z+ZD
       GO TO 2
 100   FORMAT(2(I3),3(E15.7))
 200   FORMAT(//2H N,7X1HZ,13X7HEXP(-Z),9X6HSIN(Z),10X6HCOS(Z)
      1,10X6HTAN(Z)/)
 300   FORMAT(I2,5(2XE14.7))
       END
```

APPENDIX E

FORTRAN PROGRAM FOR COMPUTING RATIONAL APPROXIMATIONS TO E(z)

Here we give a brief description of a FORTRAN program which computes the rational approximations to $E(z)$ defined by (5.11). A listing of the program follows the description of input and output data.

Let $z = re^{i\theta}$ and suppose we wish to compute the n^{th} order rational approximation to $E(z)$ as defined by (5.11) for $r_I(\Delta r)r_F$ and $\theta_I(\Delta\theta)\theta_F$ where I and F subscripts denote initial and final values, respectively.

Input: (Cards) The values r_I , r_F , Δr , θ_I , θ_F and $\Delta\theta$ are entered in this order (θ in degrees) , three numbers per card.

Output: (Typed) For each pair of values of r and θ , the n^{th} order rational approximation is printed, real part and then the imaginary part.

Switch settings: None.

```
C     RATIONAL APPROXIMATIONS TO E(Z)
      DIMENSION F1(30),F2(30),F3(30),F4(30),P1(30),
      P2(30),P3(30),P4(30)
1     READ 100,RI,RF,RD,THI,THF,THD
      CF=3.14159265358979323846264 3/180.
      DO 2 J=1,30
      F1(J)=0.0
      F2(J)=0.0
      F3(J)=0.0
      F4(J)=0.0
      P1(J)=0.0
      P2(J)=0.0
      P3(J)=0.0
2     P4(J)=0.0
      F3(1)=1.0
      F3(2)=24.0
      F3(3)=180.0
      F3(4)=480.0
      F2(1)=1.0
      F2(2)=12.0
      F2(3)=36.0
      F1(1)=1.0
      F1(2)=4.0
      P3(1)=0.0
      P3(2)=17./3.
      P3(3)=60.0
      P3(4)=480.0
      P2(1)=0.0
      P2(2)=3.0
      P2(3)=36.0
      P1(1)=0.0
      P1(2)=4.0
      N=4
3     M=N+1
      H=N
      TWH=2.*H
      A1=(H-2.)*(TWH-1.)/(H*(TWH-3.))
      A2=1.0
      A3=-A1
      B1=2.*(TWH-1.)*(H+1.)/H
      B2=-2.*(TWH-1.)*(H-3.)/H
      F4(1)=A1*F3(1)+A2*F2(1)+A3*F1(1)
      P4(1)=A1*P3(1)+A2*P2(1)+A3*P1(1)
      DO 4 J=2,M
      F4(J)=A1*F3(J)+B1*F3(J-1)+A2*F2(J)+B2*F2(J-1)+A3*F1(J)
4     P4(J)=A1*P3(J)+B1*P3(J-1)+A2*P2(J)+B2*P2(J-1)+A3*P1(J)
      DO 5 J=1,M
      F1(J)=F2(J)
      F2(J)=F3(J)
      F3(J)=F4(J)
      P1(J)=P2(J)
```

```
      P2(J)=P3(J)
5     P3(J)=P4(J)
6     RO=RI
7     R=1./RO
      TH=THI
8     PRINT 101,RO,TH
      T=TH*CF
      RE=R*COSF(T)
      RIM=-R*SINF(T)
      SRN=P4(M)*RE+P4(M-1)
      SIN=P4(M)*RIM
      SRD=F4(M)*RE+F4(M-1)
      SID=F4(M)*RIM
      DO 9 J=3,M
      L=M-J+1
      S=RE*SRN-RIM*SIN+P4(L)
      SIN=RE*SIN+RIM*SRN
      SRN=S
      S=RE*SRD-RIM*SID+F4(L)
      SID=RE*SID+RIM*SRD
9     SRD=S
      DEN=SRD*SRD+SID*SID
      QR=(SRN*SRD+SIN*SID)/DEN
      QI=(SIN*SRD-SRN*SID)/DEN
      PRINT 104,QR,QI
      TH=TH+THD
      IF (THF-TH)10,8,8
10    RO=RO+RD
      IF (RF-RO)12,7,7
12    CONTINUE
13    N=N+1
      GO TO 3
100   FORMAT (3E15.7)
101   FORMAT(E14.7,2X,E14.7)
104   FORMAT(2(E32.24))
      END
```

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